The Risk of a Mortality Catastrophe*

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Abstract

We develop a continuous-time model for analyzing and valuing catastrophe mortality contingent claims based on stochastic modeling of the force of mortality. We derive parameter estimates from a 105-year time series of U.S. population mortality data using a simulated maximum likelihood approach based on a particle filter. Relying on the resulting parameters, we calculate loss profiles for a representative catastrophe mortality transaction and compare them to the “official” loss profiles that are provided by the issuers to investors and rating agencies. We find that although the loss profiles are subject to great uncertainties, the official figures fall significantly below the corresponding risk statistics based on our model. In particular, we find that the annualized incidence probability of a mortality catastrophe, defined as a 15% increase in aggregated mortality probabilities, is about 1.4%—compared to about 0.1% according to the official loss profiles.

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The ongoing Ebola epidemic in West Africa and associated scares in developed countries suggest that the incidence of a mortality catastrophe in the U.S. may not be as inconceivable as many would have thought just a year ago. For instance, the World Health Organization (WHO) declared the Ebola virus disease a Public Health Emergency of International Concern in August 2014, in part since a coordinated international response was deemed “essential to stop [...] the international spread of Ebola” (WHO, 2014). This paper substantiates the point that this is also true from a long-term historical perspective. More precisely, we develop a model for analyzing and valuing catastrophe mortality contingent financial claims and estimate it based on a long time series of historical (U.S.) mortality data. We then calculate loss profiles for a representative catastrophe (CAT) mortality transaction and compare them to the “official” loss profiles provided in the offering circular. We find that although the risk profiles are subject to great uncertainties, the official statistics fall an order of magnitude below the spectrum of suitable estimates based on our model. Specifically, we find that the three-year incidence probability for a mortality catastrophe where aggregated mortality rates increase by more than 15% is roughly 4.1% (or 1.4% on an annualized basis)—compared to about 0.3% (0.1%) according to the official statistics.

The first part of the paper explains in detail the structure of CAT Mortality (CATM) securitization transactions and provides a survey of past transactions. Subsequently, we introduce our model. While there is an array of different stochastic mortality models available—most famously, the widely applied model by Lee and Carter (1992)—none of them satisfy all requirements we deem necessary in this context: 1) The model ought to comprise a “catastrophe” component; 2) it ought to allow for a long-term fit for the consideration of periods with mortality catastrophes, particularly the “Spanish Flu” of 1918/1919; and 3) it should be tractable.

A variety of models satisfy one of these properties but there are no models available that meet all three. Our specification consists of three parts: 1) A catastrophe component governed by
a non-negative, affine jump process; 2) a deterministic component possibly entailing a temporary trend to capture, depending on the situation, temporary mortality effects or a selection effect; and 3) an age-dependent baseline component, which models regular fluctuations of mortality over time, driven by an affine diffusion process. With at most eleven parameters, our model displays all the above properties. In particular, due to its affine structure, it exhibits a high degree of analytical tractability. For example, survival probabilities and, hence, actuarial present values can be calculated efficiently via the solution of simple ordinary differential equations. Thus, the model offers financial analysts and investors a convenient workhorse for evaluating CATM securities.

We derive parameter estimates based on a long time series of realized U.S. population mortality (1901-2005). More specifically, we rely on a maximization of the simulated likelihood using a particle filter for the likelihood estimation following ideas from Pitt (2002). Similar algorithms have also been used in other studies such as in Duan and Fulop (2009) for estimating a structural credit risk model or in Fernández-Villaverde and Rubio-Ramírez (2007) for estimating a business cycle model. The likelihood-based specification analysis reveals that for accurately describing U.S. male mortality data during the last century, a temporary linear trend for the first half of the 20th century and the catastrophe component are necessary. However, due to the low frequency of catastrophic events, the standard errors of the jump process parameters are rather large.

Relying on the parameter estimates, we calculate loss profiles for a representative CATM transaction, the Tartan issue by Scottish Re/Goldman Sachs in 2006. Our results differ tremendously from the official loss profiles and show a considerably higher risk. More precisely, while the provided statistics for the lower risk tranche indicate a default probability of less than one percent, we obtain a default probability of close to eight percent. The intuition why our result far exceeds one percent is quite straightforward. Our estimation reveals that there were several (at least three) mortality catastrophes in the 20th century, and one of them—the
1918 influenza pandemic—would have triggered default and possibly wiped out the tranche. For the catastrophe component to allow for such an observation, commensurate parameter levels are necessary such that incidence probabilities by far exceed the given statistics.

This finding is robust to estimation errors. We calculate loss profiles for stressed parameter sets, where we set each parameter at its lower and upper 90% confidence bound. While we find that errors in parameters translate to considerable uncertainties in the loss statistics, they still well exceed the official figures in each case. For instance, the most significant reduction of default probability occurs for errors in the catastrophe frequency, and here we still obtain a default probability of more than six percent for the low-risk tranche. However, the finding is not robust to a variation of the length of the underlying time series. When only considering mortality data for the latter half of the century, we do not find evidence for mortality catastrophes so that default probabilities are essentially nil—although the complete absence of the potential for a mortality catastrophe of course appears dubious.

**Related Literature and Organization of the Paper**

There are a number of papers that estimate loss profiles of credit-risky securities and show that it is possible to improve upon prevalent industry models, e.g. by accounting for stochastic volatility or unobservable risk factors (Das et al., 2007; Duffie et al., 2009; Giesecke and Kim, 2011; Gordy and Willemann, 2012). However, to our knowledge, there exist few corresponding contributions on insurance-linked securities (for exceptions in the context of earthquake insurance, see Härdle and Cabrera (2010) and references therein). This may not be surprising for conventional CAT Bonds or CAT derivatives, since such an analysis may prove difficult as the modeling of natural perils generally is very complex (see Froot (2001) for more details on the market for CAT securities in general).

However, this argument does not apply for CATM bonds, where it in principle seems feasible to devise a comprehensible and traceable model. In fact, a growing literature has
formed in statistics and actuarial science that considers *stochastic mortality modeling* (see e.g. Cairns, Blake, and Dowd (2008) for a survey). We contribute to this literature by presenting a tractable affine mortality model (Biffis, 2005; Gouriéroux and Monfort, 2008; Dahl, Melchior, and Møller, 2008; Bauer, Benth, and Kiesel, 2012) that takes into account mortality catastrophes and fits mortality experience well over a long time period. A few papers in this literature consider CATM securities, albeit with a focus on pricing. Some of these papers propose generalizations of the Lee and Carter (1992) model with non-Gaussian components in the time-varying index (Chen and Cox, 2009; Deng, Brockett, and MacMinn, 2012). However, resulting models not only entail questionable assumptions in view of the impact of a mortality catastrophe (cf. Section 3), but the utilized two-stage estimation approach is also problematic when departing from the normality assumption (Girosi and King, 2007). Other papers choose to model the aggregated mortality index (Lin and Cox, 2008; Lin, Liu, and Yu, 2013), but clearly such an approach is limited to a particular transaction—and therefore does not offer the possibility for analysts to evaluate and compare different securities.¹

The remainder of the paper is structured as follows: Section 2 gives an overview of CATM bonds and their market; Section 3 introduces our affine mortality model for their analysis; Section 4 details the estimation approach and presents a specification analysis as well as parameter estimates for 105 years of mortality data; Section 5 depicts the loss profiles for our representative CATM securitization transaction; Section 6 provides an alternative estimation of the model based on the abbreviated time series; and, finally, Section 7 concludes.

## 2 THE MARKET FOR CATM BONDS

Within CATM securitizations, an insurer or a reinsurer transfers catastrophic mortality risk from their liability side to the capital market by means of CATM bonds. This risk arises from the exposure to, for instance, severe pandemics or natural catastrophes. Traditionally,

¹Note that some of these articles post-date earlier versions of this paper.
these risks have been shared between insurers and reinsurers via reinsurance or retrocession. However, in contrast to these classical approaches, securitization bears several advantages (Cowley and Cummins, 2005; Niehaus, 2002).

Until the end of 2013, there have been ten public transactions. While they differ in their coverage area, credit ratings, or spread levels, the basic structure is the same. A certain underlying mortality index based on the mortality experience in the coverage area is defined. If this index exceeds a certain level, the bond is triggered, i.e. the investors start losing their principal. In what follows, we detail the structure of one representative example, namely the third transaction: The Tartan transaction arranged by Goldman Sachs for the reinsurer Scottish Re (Linfoot, 2007). An overview of the other transactions is presented in Table 6 in Appendix B.

Figure 1 illustrates the structure of the Tartan transaction. SALIC (Scottish Annuity & Life Insurance Company (Cayman) Ltd.), a member of the Scottish Re Group Ltd., entered into a counter-party agreement with the special purpose vehicle Tartan Capital Ltd. (Tartan). Under this agreement, Tartan would have been obligated to make payments to SALIC in case a certain index had been triggered. In return, SALIC agreed to pay Tartan a certain fixed amount
quarterly. In order to raise funds for the conditional payments to SALIC, Tartan issued and sold bonds to capital market investors. The proceeds were used to buy eligible securities, which acted as collateral. As these collateral assets could have decreased in market value, Tartan went into a swap agreement with Goldman Sachs, who also structured the deal. In return for the variable investment income from the collateral account, Goldman Sachs agreed to pay the 3-month LIBOR minus a fee of 10 basis points (bps).

Tartan issued two series of 3-year notes: A $75 million (mn) (Class A) and an $80 mn (Class B) tranche with different risk exposures. Within the Class A notes, both interest payments and the investors’ principal were guaranteed by the monoline insurer Financial Guaranty Insurance Co. (FGIC). Therefore, the primary risk that investors in the Class A notes had to face is credit risk. In return for the guarantee, FGIC received a premium from Tartan. Class B investors, on the other hand, were directly exposed to catastrophe mortality risk, i.e. they would have lost interest and principal in case of a trigger event.

The bonds and thus the payment to SALIC would have been triggered if a well-defined parametric index had exceeded a certain level. This so-called combined mortality index is contingent on the mortality experience of certain populations, and the objective is to design it such that the actual catastrophe mortality exposure of the protection buyer is reflected as accurately as possible. Within the Tartan transaction, this index was solely based on U.S. population mortality. For each relevant point in time (calendar year) \( t \), the mortality rates, i.e. the probabilities to decease within the following year, for certain partitions of the whole population as reported by the Center for Disease Control and Prevention (CDC), are aggregated to determine a weighted population death rate \( \hat{q}_t \):

\[
\hat{q}_t = \sum_x \left[ \omega_{x,m} \hat{q}_{m,x,t} + \omega_{x,f} \hat{q}_{f,x,t} \right],
\]

where \( \hat{q}_{m,x,t} \) and \( \hat{q}_{f,x,t} \) are the mortality rates for age group \( x \) in calendar year \( t \) for males and females, respectively, and \( \omega_{x,m} \) and \( \omega_{x,f} \) are the weights applied to the corresponding mortality
The risk of a mortality catastrophe displays the weights for the Tartan transaction.

<table>
<thead>
<tr>
<th>Age Groups ($x$)</th>
<th>Age Weights: Male ($\omega_{x,m}$)</th>
<th>Age Weights: Female ($\omega_{x,f}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–4</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>5–14</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>15–24</td>
<td>0.4%</td>
<td>0.4%</td>
</tr>
<tr>
<td>25–34</td>
<td>8.2%</td>
<td>6.1%</td>
</tr>
<tr>
<td>35–44</td>
<td>26.0%</td>
<td>12.7%</td>
</tr>
<tr>
<td>45–54</td>
<td>21.4%</td>
<td>7.8%</td>
</tr>
<tr>
<td>55–64</td>
<td>9.8%</td>
<td>2.7%</td>
</tr>
<tr>
<td>65–74</td>
<td>2.3%</td>
<td>0.8%</td>
</tr>
<tr>
<td>75–84</td>
<td>0.6%</td>
<td>0.4%</td>
</tr>
<tr>
<td>84+</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Total</td>
<td>68.9%</td>
<td>31.1%</td>
</tr>
</tbody>
</table>

Table 1: Gender and age weights for the Tartan transaction (Source: Linfoot (2007)).

Now the actual index at time $t$, $i_t$, is derived from the weighted population death rates at times $t$ and $t - 1$ as well as the weighted population death rates for the reference years 2004 and 2005, which are also determined according to Equation (1), by the relationship

$$i_t = \frac{1}{2} \left( \hat{q}_t + \hat{q}_{t-1} \right) \frac{1}{2} \left( \hat{q}_{2005} + \hat{q}_{2004} \right).$$

(2)

Since the index relies on the experience of two consecutive years and since Tartan issued bonds with a three year tenor, there are only two dates at which the index was calculated and at which the principal may have been reduced due to a potential catastrophic event: At the end of 2007 for the years 2006 and 2007, and at the end of 2008 for the years 2007 and 2008. In particular, this implies that investors could not lose principal in the first two years. However, the data for the index calculation will usually not be available until awhile after the respective measurement dates. Therefore, Tartan had the possibility—which they did not take advantage of—to extend the tenor of the notes up to a maximum of 30 months, but the securities would not have been able to suffer any losses due to a possible event within the extension period, and investors would have received ongoing interest payments.
Furthermore, only if the index had exceeded a certain level, the so-called trigger level or attachment point $a$, would investors have lost principal. If the index had exceeded the so-called exhaustion level or detachment point $d$, their complete principal would have been lost. For index levels between the attachment and detachment points, the loss percentage of the principal $l_t$, $t = 2007, 2008$, would have been determined as follows:

$$l_t = \min \left\{ \max \left\{ l_{t-1}, \frac{i_t - a}{d - a} \right\}, 100\% \right\},$$

(3)

where $l_{2006} = 0$. Quarterly coupons are only paid on the remaining principal. Table 2 provides the trigger and exhaustion levels as well as the interest on the Tartan notes.

<table>
<thead>
<tr>
<th></th>
<th>Class A Notes</th>
<th>Class B Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tranche Size</td>
<td>$75mn$</td>
<td>$80mn$</td>
</tr>
<tr>
<td>Term</td>
<td>3 years</td>
<td>3 years</td>
</tr>
<tr>
<td>Trigger Level</td>
<td>115%</td>
<td>110%</td>
</tr>
<tr>
<td>Exhaustion Level</td>
<td>120%</td>
<td>115%</td>
</tr>
<tr>
<td>Coupon (bps)</td>
<td>LIBOR+19</td>
<td>LIBOR+300</td>
</tr>
<tr>
<td>Rating</td>
<td>Aaa/AAA</td>
<td>Baa3/BBB</td>
</tr>
</tbody>
</table>

Table 2: Program Summary of the notes issued by Tartan (Source: Linfoot (2007)). “Rating” refers to Rating at Issuance from Moody’s Investors Service (Moody’s) and Standard and Poor’s (S&P). Class A Notes were subsequently downgraded to S&P rating ‘A’ due to major changes in the rating of FGIC. Class B Notes were subsequently downgraded to ‘BB’ based on Scottish Re being the premium payer.

As mentioned earlier in this section, Tartan was the third public catastrophe mortality transaction. Table 6 in Appendix B provides an overview of all transactions, with details for deals until the end of 2008. In most cases, the underlying mortality index is based on the populations from several countries. In this case, the combined mortality index is defined as the weighted average over the indices from the individual countries determined according to Equation (2) with the country weights provided in Table 6. For more details on each transaction, we refer to the corresponding new issue reports or the annual ILS reviews by Lane Financial LLC (Lane and Beckwith, 2008-13).
3 ANALYSIS OF CATM BONDS: AN AFFINE MORTALITY MODEL

Aside from the arranger, rating agencies etc., so-called “risk modeling firms” play an important role in CATM securitization. They are primarily appointed to calculate loss probabilities and expected losses for the different tranches of a securitization. These loss profiles are important as investors and rating agencies base their decisions on this information (Standard and Poor’s, 2008).

The primary source of risk the securities are exposed to is catastrophic mortality risk. Credit or counter-party risks usually are small for the non-guaranteed tranches due to the fully collateralized transaction design, at least for the majority of the principal. Hence, a model for the analysis of CATM bonds must allow for the possibility of an extreme evolution of mortality corresponding to, e.g., nationwide pandemics, major terrorists attacks, or natural disasters. While such scenarios may be generated based on a causal model for the different perils, the construction of these types of model is non-trivial and requires the expertise in a large array of different fields. In particular, every specification will naturally exhibit a high degree of subjectivity and a very low degree of transparency.

A more feasible and more transparent approach to analyzing CATM bonds is to rely on a statistical/actuarial model, and this has been the foundation of the official loss profiles in all CATM transactions until the end of 2008 (Standard and Poor’s, 2008). However, the transparency of this information is also limited since, to our knowledge, few details on the underlying model are published (some details of the model structure are revealed in the offering circulars and in Beelders and Colarossi (2004), but the information is sparse).

In the academic literature, several stochastic mortality models have been proposed—most prominently, the Lee and Carter (1992) model and its derivatives. However, as indicated in the introduction, these models are not suitable for our purposes since they do not allow
for “mortality catastrophes” and since they do not provide a long-term fit necessary for the consideration of periods with mortality catastrophes. Furthermore, tractability is important in our context, so that we focus on the class of affine stochastic mortality models (see Biffis (2005), Gouriéroux and Monfort (2008), Dahl, Melchior, and Møller (2008), or Bauer, Benth, and Kiesel (2012) for details and examples of affine stochastic mortality models). We propose a model consisting of three components: 1) A catastrophe component; 2) an age-dependent baseline component; and 3) a deterministic component potentially entailing a temporary trend.

Following Lando (1998), we assume that we are given a $d$-dimensional stochastic process $X = (X_t)_{0 \leq t \leq T^*}$, which is assumed to be right-continuous with left limits (RCLL), as well as a positive, continuous function $\mu(\cdot, \cdot)$. We then define the time of death $\tau_{x_0}$ of an $x_0$-year old individual as the first jump time of a Cox-process with intensity $\mu(x_0 + t, X_t)$, i.e.

$$\tau_{x_0} = \inf \left\{ t : \int_0^t \mu(x_0 + s, X_s) \, ds \geq E \right\}, \quad (4)$$

where $E$ is a unit-exponentially distributed random variable independent of $X$ and mutually independent for different individuals. As a consequence, the force of mortality $\mu_t(x_0)$ for an $x_0 + t$ year old person at time $t$ in our setup is a function of the age, $x_0 + t$, and the realization of $X$ at time $t$.

Considering only one individual for now, let the filtrations $\mathbf{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ and $\mathbf{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be given as the augmentations of the filtrations generated by $(X_t)_{0 \leq t \leq T^*}$ and $(\mathbf{1}_{\{\tau_{x_0} \leq t\}})_{0 \leq t \leq T^*}$, respectively, and set $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. From Equation (4), we can then derive the $(T - t)$-year survival probability at time $t$ for an $x_t = x_0 + t$ year old individual as

$$T^{-t} p_{x_t}(t) = \mathbb{E} \left[ \mathbf{1}_{\{\tau_{x_0} > T\}} \left| \mathcal{G}_t, \tau_{x_0} > t \right. \right] = \mathbb{E} \left[ \exp \left\{- \int_t^T \mu(x_0 + s, X_s) \, ds \right\} \left| \mathcal{G}_t, \tau_{x_0} > t \right. \right], \quad (5)$$
and, from results of Lando (1998),

\[ \mathbb{E} \left[ \mathbf{1}_{\{\tau_{x_0} > T\}} \bigg| \mathcal{F}_t \right] = \mathbf{1}_{\{\tau_{x_0} > t\}} T - t p_{x_0 + t}(t) \]

and, particularly, \( \mathbb{E} \left[ \mathbf{1}_{\{\tau_{x_0} > T\}} \right] = T p_{x_0}(0) \). In order to specify a particular model, it is now sufficient to specify an RCLL process \( X \) as well as a function \( \mu(\cdot, \cdot) \).

I. The Catastrophe Component

Modeling the micro-dynamics of an extreme mortality event such as the unfolding of a severe pandemic or deaths related to a natural disaster is very complex. From a macro-perspective, on the other hand, all these incidences will lead to a “sudden spike” in the force of mortality, i.e. to inaccessible, transitory, positive jumps. Such a structure may be described by adding a process \( \Gamma = (\Gamma_t)_{0 \leq t \leq T^*} \) governed by a stochastic differential equation (SDE) of the form

\[ d\Gamma_t = -\kappa \Gamma_t dt + dJ_t, \quad \Gamma_0 \geq 0, \quad (6) \]

to the without-catastrophe mortality intensity, where \( \kappa > 0 \) and \( J = (J_t)_{0 \leq t \leq T^*} \) is a compound Poisson process with intensity \( \lambda \) and positive, independent \( \text{Exp}(\zeta) \)-distributed jumps, \( \zeta > 0 \). In particular, we assume a constant incidence intensity as a potential trend would be difficult to assess. Aside from the analytic appeal, parsimony and assessability are also the primary motivations for the specification of the jump size distribution.

In contrast, Chen and Cox (2009) and Deng, Brockett, and MacMinn (2012) model catastrophic events as multiplicative shocks, implying that the absolute impact of a catastrophe will be considerably larger for the elderly population. While their approach could be justified by the idea that due to weaker immune systems, the elderly population may be most affected by a possible pandemic, within the most severe pandemics of the last century, a disproportionate amount of young adults had been killed due to potentially fatal immune reactions (so-called
cytokine storms). Also, natural disasters or man-made catastrophes are not likely to affect older individuals more than the younger population, which serves as a further justification for our model choice.

II. The Age-dependent Baseline Component

Again motivated by parsimony and in view of the relatively short maturities of the securities, we restrict ourselves to one-factor models and similarly to Dahl, Melchior, and Møller (2008) propose a baseline force of mortality of the form

\[ \hat{\mu}_t(x_0) = \mu^0(x_0 + t) Y_t, \]

where \( \mu^0(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) depicts the basic shape of the mortality curve and \( Y = (Y_t)_{0 \leq t \leq T^*} \) governs mortality improvements. As for \( \mu^0(\cdot) \), to keep the number of parameters small and in view of our intended application, we consider a simple Gompertz form, i.e. \( \mu^0(x) = e^{bx} \), since the Gompertz form provides a good fit for the relevant age window between 30 and 70 (cf. Table 1). However, to provide a better fit of the mortality curve when necessary, this choice clearly is easily amendable.

Choosing \( Y \), on the other hand, is a lot more intricate since a suitable specification should provide a pertinent fit to (long-term) past observations as well as entail an apposite assumption about the future mortality trend. While Girosi and King (2008) point out that with respect to the latter, every choice will inevitably be “unverifiable,” demographic data and research suggest some stylized facts, which may serve as the foundation for building an appropriate parametric model:

(S1) The (baseline) force of mortality is governed by a positive stochastic process.

(S2) Over the past century, (period) life expectancy has exhibited a linear trend increasing at
a pace of a little under three months per year (Oeppen and Vaupel, 2002).\(^2\)

(S3) Nevertheless, most likely improvements in life expectancy will level off at some point, although researchers disagree about the target level (Olshansky, Carnes, and Désquelles, 2001; Carnes and Olshansky, 2007).

(S4) **Rectangularization:** The variability in the distribution of ages at death has steadily decreased over the last century (Olivieri, 2001; Pitacco, 2004). However, this decline has slowed down in the last decades (Wilmoth and Horiuchi, 1999).

A measure for the variability of ages at death is the so-called *demographic entropy* \(H_t\), and one can show that a relative reduction of \(\delta_t\%\) in the force of mortality across all ages at time \(t\) yields an approximate increase in life expectancy at birth of \(\delta_t H_t\%\) (Keyfitz, 1985). Hence, the observed linear trend in life expectancy (S2) together with the fast decline in demographic entropy over the earlier part of the 20\(^{th}\) century (S4) suggests increasing mortality improvements in this period (see also Lee (1992, p. 321)). A steadily decreasing or constant \(\delta_t\) as e.g. prescribed by a “simple” mean-reverting process with a constant mean reversion level \(\beta^{(2)} \geq 0\) for \(Y\)—as it is frequently assumed in the actuarial literature—will not capture this observation. In particular, our (unconstrained) estimation exercises for simple mean-reverting processes result in a negative speed of mean reversion, which further illustrates the deficiency of this choice. Nevertheless, a limit to life expectancy (S3) implies that \(Y\) will revert around a basic level, of say \(\beta^{(2)}\), from some point onwards.

Hence, we propose the following dynamics for \(Y\):

\[
\begin{align*}
dY_t &= \alpha \left( \left( Y_0 - \beta^{(2)} \right) e^{-\beta^{(1)} t} + \beta^{(2)} Y_t - Y_t \right) dt + \sigma \sqrt{Y_t} dW_t, \quad Y_0 > 0, \\
\end{align*}
\]

\(^2\)While the primary result in Oeppen and Vaupel (2002) is that “life expectancy in the record-holding country” has increased linearly over time, they also demonstrate that the same trend applies to life expectancies in individual countries (see Figure 2 in their paper).
where \((W_t)_{0 \leq t \leq T^*}\) is a one-dimensional Brownian motion and \(\alpha, \beta^{(1)}, \beta^{(2)},\) and \(\sigma\) are positive constants with \(\alpha \neq \beta^{(1)}\), which (potentially) complies with (S1)-(S4). To see this, we disregard the stochastic part—which was chosen to ensure positivity (S1)—and focus on the trend part given by the ordinary differential equation (ODE)

\[
\frac{\partial}{\partial t} y_t = \alpha \left( (Y_0 - \beta^{(2)}) e^{-\beta^{(1)} t} + \beta^{(2)} - y_t \right), \quad y_0 = Y_0.
\]  

(8)

In particular, \(y_t\) is the expected value of \(Y_t\), \(\mathbb{E}[Y_t] = y_t\). The unique solution to (8) is

\[
y_t = \beta^{(2)} + (Y_0 - \beta^{(2)}) \times \left( e^{-\beta^{(1)} t} \frac{\alpha}{\alpha - \beta^{(1)}} - e^{-\alpha t} \frac{\beta^{(1)}}{\alpha - \beta^{(1)}} \right).
\]

Thus, \(y'_t < 0, \ t \geq 0\) (S2), with \(\lim_{t \to \infty} y_t = \beta^{(2)}\) (S3). Moreover, \(y''_t < 0, \ t \in [0, t_{IP})\), if \(\alpha < \beta^{(1)}\), i.e. mortality is decreasing at an accelerating pace until the inflection point

\[
t_{IP} = \frac{\log \{ \beta^{(1)} \} - \log \{ \alpha \}}{\beta^{(1)} - \alpha},
\]

after which the pace of mortality improvements declines (S4). Of course, alternative specifications of the mean trend level \(\beta_t\) are conceivable, e.g. by allowing for a third parameter \(\beta^{(3)}\) with \(\beta^{(2)} + \beta^{(3)}\) describing the “original” trend level:

\[
\tilde{\beta}_t = \beta^{(3)} e^{-\beta^{(1)} t} + \beta^{(2)}.
\]

(9)

However, specification (7) captures the stylized facts with a small number of parameters.

III. Temporary Component

In addition to the catastrophe and the baseline component, our model comprises an (additive) deterministic component \(D = (D_t(x_0))_{0 \leq t \leq T^*}\), possibly entailing a temporary mortality trend.
For instance, we may set:

\[ D_t(x_0) = D_t = c + \gamma \times \max \{ \bar{T} - t, 0 \} = c + \gamma \times (\bar{T} - t)^+. \]  \hspace{1cm} (10)

While \( c \) is some “extrinsic” mortality level (Carnes, Olshansky, and Grahn, 1996), the purpose of the second term of \( D \) could be twofold—depending on the sign of \( \gamma \): On the one hand, it can capture the well-documented “structural break” in the mortality evolution for most developed countries resulting from a drastic decline of deaths due to infectious disease in the first half of the twentieth century or—as put by Preston (1996)—the “scientific breakthrough” of “the germ theory of disease” (see also Wilmoth (2005)). On the other hand, \( D \) may capture a temporary reduction of mortality due to a selection effect when evaluating life insurance contracts.

Thus, all-in-all, we propose the following model for the stochastic force of mortality:

\[ \mu_t(x_0) = \mu(x_0 + t, Y_t, \Gamma_t) = e^{b(x_0 + t)} Y_t + \Gamma_t + D_t(x_0), \quad \Gamma_0, Y_0 > 0. \]  \hspace{1cm} (11)

A valuable feature is its (exponential-)affine structure (Duffie, Pan, and Singleton, 2000), which enables us to provide an analytical representation of the survival properties (cf. Prop. 1 in Duffie, Pan, and Singleton (2000)):

\[ T-tP_{x_0+t}(t) = \exp \left\{ u(T-t) + v(T-t) Y_t - \frac{\Gamma_t}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right) - \frac{\lambda(T-t)}{\zeta \kappa + 1} \right\} \]
\[ \times \exp \left\{ \frac{\lambda \zeta}{\zeta \kappa + 1} \log \left[ 1 + \frac{1}{\zeta \kappa} \left( 1 - e^{-\kappa(T-t)} \right) \right] - \bar{D}_{t,T}(x_0) \right\}, \]  \hspace{1cm} (12)

where \( u \) and \( v \) satisfy the following Riccati ODEs:

\[ u'(s) = -e^{b(x_0 + T-s)} - \alpha v(s) + \frac{1}{2} \sigma^2 v^2(s), \quad v(0) = 0, \]
\[ u'(s) = v(s) \alpha \left( e^{-\beta(1)(T-s)} Y_0 + (1 - e^{-\beta(1)(T-s)}) \beta(2) \right) , \quad u(0) = 0, \]  \hspace{1cm} (13)
and

\[ \bar{D}_{t,T}(x_0) = \int_t^T D_s(x_0) \, ds. \]

Hence, we are able to calculate survival probabilities and, thus, actuarial present values by simply solving the ODEs from Equation (13).

4 ESTIMATION

The proposed mortality model exhibits latent factors and non-Gaussian innovations, and there is a large number of data (ages) available for each panel (year). Hence, the parameter estimation problem is not trivial. While there exist a variety of potentially feasible methods in the econometrical literature (see e.g. Footnote 1 in Johannes, Polson, and Stroud (2009)), the low frequency of the data and the interpretation of jumps as catastrophic mortality events render many of them inappropriate.

Here, we rely on a maximization of the simulated likelihood using Bayesian recursive estimation in the form of a particle filter for the likelihood evaluation. Our algorithm is based on the ideas in Pitt (2002) and Malik and Pitt (2011), who develop approximations of the model likelihood based on the Sampling-Importance-Resampling Particle Filter (SIR—see Gordon, Salmond, and Smith (1993)). Similar algorithms have also been used in other studies such as in Duan and Fulop (2009) for estimating a structural credit risk model or in Fernández-Villaverde and Rubio-Ramírez (2007) for estimating a business cycle model.

4.1 State-Space Formulation and Filtering Algorithm

In order to apply any recursive estimation procedure, it is necessary to represent our model equations in a state-space formulation. Here, the raw observations are given in the form of
one-year realized survival probabilities for each year \( t \in \{0, \ldots, T - 1\} \):

\[
P_{x}^{(t+1)} = \exp \left\{ - \int_{t}^{t+1} \mu (x_s, Y_s, \Gamma_s) \, ds \right\}
= \exp \left\{ - e^{b,x_{t}} \int_{t}^{t+1} Y_s e^{b(s-t)} \, ds - \int_{t}^{t+1} \Gamma_s ds - \int_{t}^{t+1} D_s ds \right\}.
\]

In particular, the observations not only depend on the end-of-year Markov states of the transition equation, \( Z_{t+1}^{(1)} = Y_{t+1} \) and \( Z_{t+1}^{(2)} = \Gamma_{t+1} \), but on the entire paths over \( (t, t + 1) \). Hence, we need to consider the four-dimensional state vector

\[
Z_{t+1} = \left( Z_{t+1}^{(1)}, Z_{t+1}^{(2)}, Z_{t+1}^{(3)}, Z_{t+1}^{(4)} \right),
\]

the transitional distribution \( Z_{t+1} | Z_t \) of which can be sampled via the solutions of the corresponding SDEs and simple quadrature rules.

For the measurement equation, we follow Gouriéroux and Monfort (2008) and consider observations (measurements) \( (y_{t+1})_{x \in X} \)

\[
y_{t+1}(x) = - \log \left\{ P_{x}^{(t+1)} \right\} = e^{b,x_{t}} Z_{t+1}^{(3)} + Z_{t+1}^{(4)} + \int_{t}^{t+1} D_s ds + \eta_{t+1}(x),
\]

where the error terms \( (\eta_{t+1}(x))_{t=0,1,\ldots,T-1;x \in X} \) are (approximately) Gaussian, independent, with zero mean and variance

\[
\text{Var} (\eta_{t+1}(x)) = (\exp \{ y_{t+1}(x) \} - 1) \bar{\eta}.
\]

Therefore, the underlying assumptions of Section 1 in Pitt (2002) are satisfied, and we may follow his ideas to determine approximations for the model log-likelihood (cf. Eq. (1.1))
in Pitt (2002)):

$$\log L(\theta) = \log \{ f_\theta (y_1, y_2, \ldots, y_T) \} = \sum_{t=1}^{T} \log \{ f_\theta (y_t | (y_1, \ldots, y_{t-1})) \},$$

where \( \theta = (\alpha, \beta^{(1)}, \beta^{(2)}, \sigma, b, k, \lambda, \zeta, c, \gamma, \bar{T}, Y_0, Z_0) \) is the parameter vector. In order to compare different parametrizations/specifications, we fix the model error to \( \bar{\eta} = 0.0005 \) in our implementation.\(^3\) In the main text, we restrict ourselves to the statement of the final algorithm for the approximation of the likelihood given a parameter vector—further explanations and derivations are provided in Appendix A:

**Algorithm 4.1.**

- \( (z_0^{(1), k}, z_0^{(2), k}) = (Y_0, \Gamma_0), k = 1, \ldots, N \) (given).

- For \( t = 0, \ldots, T - 1 \):
  
  Given samples \( (z_t^{(1), 1}, z_t^{(2), 1}), \ldots, (z_t^{(1), N}, z_t^{(2), N}) \) \( \sim (Z_t^{(1)}, Z_t^{(2)}) | (y_1, \ldots, y_t) \):

  1. For \( k = 1 : N \), sample \( z_{t+1}^k \sim Z_{t+1} | z_t^{(1), k}, z_t^{(2), k} \).

  2. For \( k = 1 : N \), calculate

     \[
     w_{t+1}^k = \prod_{x \in X} \frac{1}{\sqrt{2\pi} \left( \exp \{ y_{t+1}(x) \} - 1 \right) \bar{\eta}} \times \exp \left\{ \frac{- \left( y_{t+1}(x) - e^{bx} z_t^{(3), k} - z_t^{(4), k} - c - \gamma (\bar{T} - t - 0.5)^+ \right)^2}{2 \bar{\eta} \left( \exp \{ y_{t+1}(x) \} - 1 \right)} \right\},
     \]

     \[
     l_{t+1} = \sum_{k=1}^{N} w_{t+1}^k.
     \]

  3. For \( k = 1 : N \), draw \( i(k) \sim \text{Mult} \left( N; \frac{w_{t+1}^1}{l_{t+1}}, \ldots, \frac{w_{t+1}^N}{l_{t+1}} \right) \) and set \( z_{t+1}^k = \tilde{z}_{t+1}^{i(k)} \).

- Calculate \( \hat{\log} L(\theta) = \sum_{t=1}^{T} \log \left\{ \frac{1}{N} l_t \right\} \).

\(^3\)This choice is roughly in line with U.S. population size (by age) in 2000. Our estimation exercises show that results are relatively insensitive to moderate changes in \( \bar{\eta} \).
4.2 Data and Results

We use annual, periodic U.S.-male mortality data as available from the Human Mortality Database. More specifically, we rely on age-specific (realized) survival probabilities for ages between 20 and 81 from 1901-2005. We restrict ourselves to male mortality experience since within the combined mortality indices, male death rates are usually weighted more heavily than female death rates (see also Table 1; note that the weights need to be adjusted if only male mortality experience is considered). While including female or non-U.S. mortality data would be relatively straight-forward from a theoretical perspective, the number of necessary parameters would further increase as correlation terms would need to be considered. In view of the already high complexity of the estimation procedure, we leave the exploration of this issue for future work. As an illustration, Figure 2 shows male death probabilities for ages 20 and 50 from our data set. As pointed out in Section 3, the rates show a general downward trend while the speed of mortality improvements varies for different periods. Moreover, we observe “spikes” due to catastrophic mortality events, most noticeably the Spanish Flu in 1918; in particular, it is apparent that the relative impact differs substantially between the two considered ages as also indicated in the previous section.

For the numerical optimization of the (log-)likelihood, we use the Nelder-Mead method as available in most scientific software packages. Of course, the same seed for the random number generation was chosen for each evaluation in order to avoid discontinuities due to Monte Carlo errors. While the algorithm converges for all the considered runs, the results are sensitive to the chosen starting values of the parameters, which is due to the high dimension of the parameter vector $\theta$ and potential discontinuities of the likelihood function (see Appendix A). We accommodate this problem by choosing a large array of different starting values and an increasing number of samples. To obtain reasonable starting values for the catastrophe com-

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4Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (downloaded 02/2009).
We rely on Linfoot (2007), who reports an annual frequency of 7.4% and the severities of several specific occurrences of catastrophic mortality events, which we roughly fit to our jump size distribution. Moreover, we initially assume that each event affects the evolution of mortality for only 1-2 years. For the baseline component, we fit Gompertz curves for several representative years, and then find starting values for \((\alpha, \beta^{(1)}, \beta^{(2)}, \sigma, Y_0)\) by analyzing the corresponding univariate time series or, more specifically, the iid Gaussian innovations of the annually-discretized version prescribed by our model equation (7). Finally, we initially set \(\Gamma_0 = 0\). For many parameter choices in the relevant region of the parameter space, we initially use a sample size of \(N = 1,000\) and \(\Delta = 12\) time steps in the simulation of \(Z_t^{(3)}\), \(t = 1, \ldots, 105\). \(Z_t^{(1)}, Z_t^{(2)},\) and \(Z_t^{(4)}\) were sampled exactly according to their respective distributions. Employing the parametrizations that lead to the largest likelihood as starting vectors, we subsequently run the optimization algorithm with sizes \(N\) equaling to 25,000, 50,000, and—finally—100,000 and \(\Delta = 24\) time steps. The final parameters are displayed in Table 3 with a log-likelihood of 32,045. We find that the parameters as well as the log-likelihood are relatively stable with respect to the sample size in the last estimation step. More specifically, the resulting changes in parameter values of less than 3% on average and maximally
approximately 10% are relatively insubstantial in view of our results.

I. Catastrophe Component

\[
\begin{array}{ccc}
\kappa & \lambda & \zeta \\
2.3654 & 0.0354 & 204.36 \\
(0.0279) & (0.0055) & (26.02)
\end{array}
\]

II. Baseline Component

\[
\begin{array}{cccccc}
\alpha & \beta^{(1)} & \beta^{(2)} & \sigma & b & Y_0 \\
0.0164 & 0.0273 & 5.19E-05 & 2.62E-04 & 0.0845 & 1.49E-04 \\
(0.0002) & (0.0006) & (4.07E-07) & (2.01E-06) & (0.0001) & (1.41E-06)
\end{array}
\]

III. Temporary Component

\[
\begin{array}{ccc}
c & \gamma & T \\
4.86E-04 & 1.07E-04 & 51 \\
(2.27E-05) & (1.73E-06) & (na)
\end{array}
\]

Table 3: Estimated parameters for the full model with jumps (model MLJ), U.S. male mortality data 1901-2005. Estimation results are based on 100,000 particles. Standard errors for each parameter are shown in parentheses.

In addition to the “full” model labeled MLJ (Makeham, Long period, with Jumps), we also fit alternative model versions relying on an analogous estimation procedure. More specifically, we estimate a version that consists solely of the baseline component labeled GLC (Gompertz, Long period, Continuous; log-likelihood 30,171); a model version that entails the baseline as well as the age-independent temporary component labeled MLC (Makeham, Long period, Continuous; log-likelihood 31,955); and a model version including the baseline and the jump (catastrophe) component labeled GLJ (Gompertz, Long period, with Jumps; log-likelihood 31,860). The corresponding parameter estimates are provided in Table 7 in Appendix B.

Figure 3 depicts the specification analysis. Due to the partially nested model structure, it is possible to compare the models using a simple likelihood ratio test. The values of the resulting test statistics (LR) are also denoted in Figure 3. They provide overwhelming evidence in favor of the more complex model at each stage: The \( p \)-value is essentially zero for all comparisons.
The risk of a mortality catastrophe

Figure 3: Comparison of the model specifications, long period.

To further illustrate the estimation and specification results, Figure 4 displays the expected values of the model states $Z_t^{(1)}/Z_t^{(3)}$ and $Z_t^{(2)}/Z_t^{(4)}$ corresponding to the baseline and the catastrophe component, respectively. We notice that in the GLC model much of the mortality improvement in the first half of the century due to the eradication of infectious disease—which in the MLC model is captured by the temporary component—is included in the baseline component. This not only leads to a considerably difference in the likelihoods of the models, but also yields substantial differences in the parametrization of the baseline component in the GLC model relative to all three other specifications (cf. Table 7 in Appendix B). In contrast, within the GLJ model, this effect is captured by the jump component. As a consequence, in this case the jump intensity $\lambda$ is considerably greater and the mean reversion level is considerably smaller than for the “full” (MLJ) model. It is worth noting that the results are very similar when $\Gamma_0$ is set to zero, the primary difference being a smaller likelihood due to large jumps in the first year.

The expected states for the full model are plotted in Figure 5. While the baseline com-
Figure 4: Expected values of the GLC, MLC, and GLJ specification; long time period. Expected values are based on $N = 500,000$ particles and $\Delta = 48$ time steps.
ponent looks very similar to the MLC and the GLJ models, the catastrophe component now shows the anticipated “spikes.” In addition to the large spike due to the Spanish flu in 1918-1919, we observe several other “mortality catastrophes.” While some of these just appear to be due to an unfavorable development of several “regular” causes of death—such as an extraordinary amount of pneumonia and tuberculosis deaths in 1907 (The Insurance Press, 1909) or the “mortality peak” in 1936/1937 (Tapia Granadas and Diez Roux, 2009)—others are associated with disastrous events. For instance, an increased death rate in the early 1940s and the late 1960s due to casualties from World War II and the Vietnam War, respectively, are noticeable.

Since much of the mortality improvement over the first half of the twentieth century is caught by the temporary component, the improvements associated with the baseline component are most pronounced in later decades resulting in an inflection point $t_{IP} \approx 47$, i.e. increasing mortality improvements in the baseline component until the middle of the century.

The ultimate mean reversion level is $\beta^{(2)} = 5.2 \times 10^{-5}$ implying a limiting male cohort life expectancy

$$
\dot{e}_{0} (\infty) = \int_{0}^{\infty} \exp \left\{ - \int_{0}^{t} e^{bs} \beta^{(2)} + c \, ds \right\} \, dt
$$

of a little under 80 years, which is roughly in line with “a realist view of [...] future mortality” (Carnes and Olshansky, 2007). More precisely, Carnes and Olshansky (2007) conclude that

---

**Figure 5:** Expected values of the MLJ model; long time period. Expected values are based on $N = 500,000$ particles and $\Delta = 48$ time steps.
absent significant advances in the control of aging, life expectancy for the entire population is unlikely to exceed 85 years, which would be—ceteris paribus—equivalent to a complete elimination of cancer and heart disease.

As described in Hamilton (1994), a common approach to determine standard errors within state-space models is to rely on the inverse of the hessian of the log-likelihood function at the MLE. Here, we choose a slight modification by considering each variable separately. The reasons are numerical challenges when evaluating the hessian off the diagonal. Nonetheless, this approach allows us to calculate confidence bounds for each parameter.

5 LOSS PROFILES

For rating agencies and investors in asset-backed securities, the probability of default (PD), i.e. the probability that the investors’ principal will be reduced, and the expected loss (EL) of principal are important statistics. Moreover, the spread level corresponding to an “actuarially fair” contract in the sense that the sum of the expected discounted cash flows equals zero (without consideration of a risk premium) may be of interest to investors and issuers. We use Monte Carlo simulations to derive the loss profiles as well as the actuarially fair spread levels for the two tranches of the Tartan deal. Table 4 displays the results. Note that for the calculation of the loss profiles, only mortality risk is considered, i.e. we disregard credit risk. Similarly, for the calculation of the spread levels, we rely on risk-free interest rates as implied by the U.S. treasury yield curve from January 2006.\(^5\) The initial model states \(\Gamma_0\) and \(Y_0\) are sampled from the (empirical) posterior distribution resulting from the estimation procedure.

All results clearly reflect the increased risk exposure of the lower Class B notes in comparison to the more senior Class A notes. Most strikingly, the calculated statistics by far exceed the official risk measures. Rather than expected losses of 0.54% for the riskier tranche and

\(^5\)Yield curve data downloaded from FRED (Federal Reserve Economic Data) provided by the St. Louis Federal Reserve Bank (www.research.stlouisfed.org).
Table 4: Loss profiles as implied by our model and as reported by the risk modeling firm. Loss profiles are calculated based on 500,000 Monte Carlo simulations.

0.16% for the less risky tranche as within the official loss profiles, we obtain expected loss levels of 5.57% and 3.44%, respectively—so the results differ by an order of magnitude. A part of this deviation may be explained by biases within our calculation. For instance, as indicated above we solely consider the male U.S. population even though the combined mortality index within the Tartan deal is based on the entire population. Resulting diversification effects as well as a potentially reduced exposure of female lives to e.g. wars or major terrorist attacks may lead to a reduction of the risk levels. However, it appears very implausible that these biases can explain the huge gap between our calculated and the reported figures. Note that although the calculated risk levels are far greater than reported, the market spread level for tranche B (300bps) still exceeds the “actuarily fair” premium (182bps). The question of whether this spread sufficiently compensates investors for the adopted risk is beyond the scope of this paper and is left for future research.

To analyze the robustness of this result, we also calculate loss profiles for stressed parameter sets, where we set each parameter at its lower and upper 90% confidence level. Table 5 displays the results for Tartan tranche B. We find that errors in the parameters translate to considerable uncertainties in the loss statistics. The influence is particularly pronounced for the Gompertz parameter $b$—although the standard error appears rather small in this case—and
for the catastrophe parameters $\lambda$ and $\zeta$. The latter observation is no surprise. Due to the small number of catastrophic mortality events, an accurate estimation of the catastrophe parameters is difficult. However, despite the considerable reduction of the default probability due to potential parameter errors, the risk statistics still by far exceed the officially reported figures. For instance, the difference in expected loss levels for the low catastrophe frequency ($\lambda$ low) still exceeds the reported profiles by a factor of eight.

6 ESTIMATION FOR AN ABBREVIATED TIME SERIES

In addition to estimating the model based on all data from the last century, we also fit our model to a reduced data set from 1950 to 2005. More precisely, we consider three alternative model versions: A model that entails the baseline component as well as the age-independent constant labeled MSC (Makeham, Short period, Continuous; log-likelihood 17,593); a version including both the baseline and the jump component labeled GSJ (Gompertz, Short period, with Jumps; log-likelihood 17,592); and a full model version, which contains all three components labeled MSJ (Makeham, Short period, with Jumps; log-likelihood 17,593). The corresponding parameter estimates are also collected in Table 7 in Appendix B and Figure 6 illustrates the specification analysis.

![Figure 6: Comparison of the specifications, short period.](image)
### Table 5: Sensitivity analyses of the loss profiles based on 90% confidence bounds for each parameter. Loss profiles are calculated based on 500,000 Monte Carlo simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PD(%)</th>
<th>EL(%)</th>
<th>Spread(bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cl. B Tranche (110%-115%)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$ low (0.0161)</td>
<td>8.03</td>
<td>5.61</td>
<td>183</td>
</tr>
<tr>
<td>$\alpha$ high (0.0168)</td>
<td>7.94</td>
<td>5.58</td>
<td>182</td>
</tr>
<tr>
<td>$\beta^{(1)}$ low (0.0263)</td>
<td>8.03</td>
<td>5.60</td>
<td>183</td>
</tr>
<tr>
<td>$\beta^{(1)}$ high (0.0282)</td>
<td>7.93</td>
<td>5.53</td>
<td>181</td>
</tr>
<tr>
<td>$\beta^{(2)}$ low (5.13E-05)</td>
<td>7.94</td>
<td>5.57</td>
<td>182</td>
</tr>
<tr>
<td>$\beta^{(2)}$ high (5.26E-05)</td>
<td>8.07</td>
<td>7.93</td>
<td>184</td>
</tr>
<tr>
<td>$\sigma$ low (2.59E-04)</td>
<td>8.00</td>
<td>5.60</td>
<td>183</td>
</tr>
<tr>
<td>$\sigma$ high (2.66E-04)</td>
<td>8.14</td>
<td>5.66</td>
<td>185</td>
</tr>
<tr>
<td>$b$ low (0.0843)</td>
<td>6.61</td>
<td>4.90</td>
<td>160</td>
</tr>
<tr>
<td>$b$ high (0.0847)</td>
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<td>213</td>
</tr>
<tr>
<td>$\kappa$ low (2.3196)</td>
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<td>4.36</td>
<td>142</td>
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<tr>
<td>$\lambda$ high (0.0444)</td>
<td>9.33</td>
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<td>220</td>
</tr>
<tr>
<td>$\zeta$ low (161.57)</td>
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<td>6.29</td>
<td>206</td>
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<tr>
<td>$\zeta$ high (247.16)</td>
<td>7.39</td>
<td>4.97</td>
<td>162</td>
</tr>
<tr>
<td>Reported</td>
<td>0.88</td>
<td>0.54</td>
<td>300</td>
</tr>
</tbody>
</table>

In contrast to the estimation for the long time series, here a (nested) likelihood ratio test results in the more parsimonious MSC model being far from rejected, while there is some evidence for the MSJ over the GSJ model. Note that both the GSJ and the MSC model dominate a model version solely including the baseline component. For the cross model comparison between the MSC and the GSJ model, a Vuong test clearly favors the MSC model due to a
reduced number of parameters. Figure 7 provides the intuition for this observation. It displays the expected states for the baseline component within the MSC model (left panel) and the jump component within the GSJ model (right panel). The latter suggests that the jump component essentially mimics an (almost) horizontal line, which can be more appositely—and more parsimoniously—captured by a simple constant as featured within the MSC model. The baseline component (Figure 7(a)), on the other hand, now shows slightly increasing mortality improvements until the late 1970s as also suggested by the inflection point of $t_{IP} \approx 29$. The resulting ultimate cohort life expectancy is $e_0(\infty) \approx 86$ years, which is even more in tune with the “realist” view from the demographical literature.

![Figure 7: Expected values of the baseline component of the MSC model (left) and the catastrophe component of the GSC model (right); short time period. Expected values are based on $N = 500,000$ particles and $\Delta = 48$ time steps.](image)

Figure 7: Expected values of the baseline component of the MSC model (left) and the catastrophe component of the GSC model (right); short time period. Expected values are based on $N = 500,000$ particles and $\Delta = 48$ time steps.

Since the model based on the reduced period does not feature any mortality catastrophes, the resulting loss profiles suggest that the default probability is essentially zero for both tranches. However, the complete absence of the potential for a mortality catastrophe appears dubious. While the improved health infrastructure, the existence of and improvements in pandemic preparedness plans, and relatively recent advances in virology may have helped to avert mortality catastrophes in the second half of the twentieth century, the absence of a severe pandemic—or a man-made disaster with similar consequences for mortality—could also be attributed to sheer chance, and with the relatively low estimate for the catastrophe frequency
based on the long time series this is not unlikely in the context of our model. In fact, even the probability for no catastrophe in an 80-year period exceeds five percent. Furthermore, the threat of new infectious diseases as well as the reemergence of infectious diseases that were thought to have been eradicated (Olshansky et al., 1997) as well as industrialized livestock husbandry and other conditions fostering mutations of viruses likely have increased in recent decades, so that it is even possible that the situation has worsened. Hence, while it is impossible to conclude that the loss profiles derived from our model are “accurate” per se, the official statistics certainly appear overly optimistic.

7 CONCLUSION

Catastrophe Mortality Bonds are a recent capital market innovation providing insurers and reinsurers with the possibility to transfer catastrophe mortality risk off their balance sheets to capital markets. We introduce a continuous-time stochastic mortality model for analyzing these securities. With at most eleven parameters, our model displays an array of basic features that are in line with demographic data and research, and it exhibits jumps that are structurally coherent with catastrophic mortality events observed in the past century. Moreover, due to its affine structure, it offers a high degree of analytical tractability. In particular, survival probabilities and, hence, actuarial present values can be calculated very efficiently via the solution of simple ordinary differential equations. We derive simulated maximum likelihood estimates from time series of U.S. population mortality data. Relying on the resulting parameters, we calculate loss profiles for a representative catastrophe mortality transaction and compare them to the official loss profiles that are provided by the issuers to investors and rating agencies.

Our key finding is that although the loss profiles are subject to great uncertainties, the official figures fall significantly below the corresponding risk statistics based on our model. In particular, we find that the incidence probability of a mortality catastrophe—defined as a 15%
increase in aggregated mortality probabilities—is more than one percent on an annualized basis.

It is important to note that this paper is solely concerned with the question of whether the loss profiles provided by information intermediaries properly reflect the risk in the securities. The question of whether the spread (300 basis points for the more risky tranche in the transaction we consider) sufficiently compensates investors for the adopted risk is beyond the scope of this paper but presents an interesting avenue for future research.

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REFERENCES


APPENDIX

A Simulated Maximum Likelihood Estimation via a Particle Filter

As explained in Section 4.1, our model can be represented in a state-space formulation with four-dimensional state vector $Z_t = \left(z_t^{(1)}, z_t^{(2)}, z_t^{(3)}, z_t^{(4)} \right)$, the transitional distribution $f_\theta (Z_{t+1} | Z_t)$ of which can be sampled via simulation.

We are concerned with the calculation/estimation of the (log-)likelihood

$$\log L(\theta) = \log \left\{ f_\theta (y_1, \ldots, y_T) \right\}$$

i.e. we need to evaluate/estimate

$$f_\theta (y_t | (y_1, \ldots, y_{t-1})) = \int f_\theta (y_t | z_t) \times f_\theta (z_t | (y_1, \ldots, y_{t-1})) \, dz_t = E \left[ f_\theta (y_t | Z_t) | (y_1, \ldots, y_{t-1}) \right]$$

for all $t$. Since $f_\theta (y_t | \cdot)$ is known by the assumption on the error terms, it suffices to obtain samples $\tilde{z}_t^1, \ldots, \tilde{z}_t^N \sim f_\theta (z_t | (y_1, \ldots, y_{t-1}))$, because then we can obtain the straight-forward estimate

$$\hat{f}_\theta (y_t | (y_1, \ldots, y_{t-1})) = \frac{1}{N} \sum_{k=1}^N f_\theta (y_t | \tilde{z}_t^k) = \frac{1}{N} \sum_{k=1}^N \prod_{x \in X} f_\theta (y_t(x) | \tilde{z}_t^k)$$

$$\Rightarrow \hat{\log L}(\theta) = \sum_{t=1}^T \log \left\{ \frac{1}{N} \sum_{k=1}^N w_t^{(k)} (\theta) \right\}.$$

Samples from $f_\theta (z_t | (y_1, \ldots, y_{t-1}))$ can now be drawn via a particle filter, which provides a method to draw samples $z_{t-1}^1, \ldots, z_{t-1}^N \sim f_\theta (z_{t-1} | (y_1, \ldots, y_{t-1}))$: Simply sample $\tilde{z}_t^k \sim f(\tilde{z}_t^k | z_{t-1}^k)$ since

$$f_\theta (z_t | (y_1, \ldots, y_{t-1})) = \int f_\theta (z_t | z_{t-1}, (y_1, \ldots, y_{t-1})) \times f_\theta (z_{t-1} | (y_1, \ldots, y_{t-1})) \, dz_{t-1}$$

$$= \int f_\theta (z_t | z_{t-1}) \times f_\theta (z_{t-1} | (y_1, \ldots, y_{t-1})) \, dz_{t-1}.$$
We rely on the so-called Sampling-Importance-Resampling Particle filter (SIR—Gordon, Salmond, and Smith (1993)), the basic algorithm of which reads as follows (Malik and Pitt, 2011) (for notational convenience, we omit the $\theta$-subscripts):

**Algorithm A.1.**

1. **Sample** \( \left( (z_0^{(1)}, z_0^{(2)}), \ldots, (z_N^{(1)}, z_N^{(2)}) \right) \sim f_{y_0, \gamma_0}(y, \gamma) \) (given).

2. **For** \( t = 0, \ldots, T - 1 \):
   
   Given samples \( \left( (z_t^{(1)}, z_t^{(2)}), \ldots, (z_t^{(1)}, z_t^{(2)}) \right) \sim \left( Z_t^{(1)}, Z_t^{(2)} \right) \mid (y_1, \ldots, y_t) \):
   
   1. For \( k = 1 : N \), sample \( \tilde{z}_{t+1}^k \sim Z_{t+1} \mid z_t^{(1), k}, z_t^{(2), k} \).
   
   2. For \( k = 1 : N \), calculate
      \[
      w_{t+1}^k = f(y_{t+1} \mid \tilde{z}_{t+1}^k), \quad l_{t+1} = \sum_{k=1}^N w_{t+1}^k.
      \]
   
   3. For \( k = 1 : N \), sample
      \[
      z_{t+1}^k \sim \sum_{i=1}^N \frac{w_{t+1}^i}{l_{t+1}} \times \delta(\cdot - \tilde{z}_{t+1}^i),
      \]
      where \( \delta(\cdot) \) is the Dirac-Delta function.

As pointed out by Pitt (2002) and Malik and Pitt (2011), a potential problem with Algorithm A.1 are possible discontinuities of the likelihood function. This is simply due to the fact that in step 3., we are sampling from an empirical distribution function, which itself is discontinuous (see Malik and Pitt (2011) for a detailed discussion). As described in the main text, we accommodate this issue by considering a large array of different starting values. Moreover, the SIR algorithm is known to suffer from a problem that is usually referred to as sample impoverishment (see Johannes, Polson, and Stroud (2009) for details), especially if the number of samples is small. Therefore, while the first estimation step is based on a limited number of 1,000 particles, subsequent estimations are performed based on relatively large sample sizes.

### B Additional Tables
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<th>Vita Capital Ltd.</th>
<th>Vita Capital II Ltd.</th>
<th>Tartan Capital Ltd.</th>
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<td>Swiss Re</td>
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<td>Protection for</td>
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<td>Swiss Re</td>
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<td>US 62.5%, UK 17.5%, D 7.5%, J 7.5%, CAN 5%</td>
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<td>US 62.5%, UK 17.5%, J 7.5%, CAN 5%</td>
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6 The tranches marked with * are guaranteed by monoline insurers. Most of these tranches were downgraded in 2008 due to trouble of the guarantors.

7 Rating at Issuance from Moody’s / S&P – the ratings marked with ** were upgraded by S&P.
### Parametrizations

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<th>Mod.</th>
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<th>$b$</th>
<th>$\kappa$</th>
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<th>$\zeta$</th>
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Table 7: Parameter estimates for the different models. The parameters marked with * were set and not estimated.