Risk and Valuation of Premium Payment Options in Participating Life Insurance Contracts

Daniel Bauer†
Department of Risk Management and Insurance, Georgia State University
35 Broad Street, 11th Floor; Atlanta, GA 30303
Phone: (404)-413-7490, Fax: (404)-413-7499
Email: dbauer@gsu.edu

Richard D. Phillips
Department of Risk Management and Insurance, Georgia State University
Email: rphillips@gsu.edu

Adam L. Speight
Email: aspeight@gmail.com


*An earlier version of this paper was entitled “Comment on the Paper ‘Assessing the Risk Potential of Premium Payment Options in Participating Life Insurance Contracts’ by N. Gatzert and H. Schmeiser”.
†Corresponding author.
Risk and Valuation of Premium Payment Options in Participating Life Insurance Contracts

Abstract

In this paper, we reconsider the risks associated with paid-up and resumption premium options commonly found in participating life insurance contracts as studied by Gatzert and Schmeiser (2008). We do so by modifying the assumption underlying their methodology that information about policyholders exercising the options during the lifetime of the contract is not revealed to the insurance company until maturity. We show the results of their risk assessment exercise change significantly under the more realistic notion that the company knows about policy endorsements as they occur: in this case the premium options do not expose the insurer to additional risk. However, by considering the premium options in a model setting where the company’s reserve situation affects the distribution to policyholders – which is more suitable for many markets – we reconfirm Gatzert and Schmeiser’s primary proposition that these options may place the insurer at significant risk.

1 Introduction

Participating life insurance contracts, i.e. those that provide a minimum interest rate guarantee plus the possibility to participate in the earnings of the insurer, are popular in many markets around the world and have been widely analyzed in the academic literature (see e.g. Grosen and Jørgensen (2000), Miltersen and Persson (2003), or Bauer et al. (2006)). However, common features in these contracts not typically studied are premium payment options. An exception is the paper by Gatzert and Schmeiser (2008) (GS in what follows) in which the authors model a paid-up option that allows the policyholder to discontinue future premium payments at policy anniversaries and a resumption option that provides the policyholder the possibility to resume making premium payments at a policy anniversary on a contract they had previously stopped. The valuation methodology GS adopt they describe as “not based on assumptions about a particular policyholder’s exercise strategy” but instead “assesses the risk potential from the insurer’s viewpoint.” Their analysis suggests the paid-up and resumption options have positive value and that they place the insurer at significant risk.
We have two purposes for this paper. The first is to reconsider the informational assumption made by GS that requires policyholders exercising their options to be private information not revealed to the insurance company until policy expiry. We argue this assumption is unconventional and leads to unreasonably large estimates of the potential risk exposure. In this paper, we assume company managers learn about policy endorsements as they occur and demonstrate this modification changes the risk assessment exercise significantly as the premium options in the GS framework under the less restrictive information structure pose no risk to the insurer. Specifically, the cheapest super-replication strategy cast in the form of an optimal stopping problem yields option values equal to zero under any exercise behavior – even the “lucky” or prescient ones considered by GS.¹

Our second purpose is to show this conclusion that paid-up and resumption premium options are worthless is the result of the specific contract GS consider and is therefore not a generalizable conclusion. We do so by deriving (worst-case) premium option values when we redefine the participating contract using a variant of the earnings distribution scheme described in Kling et al. (2007) and Bauer et al. (2006).

We conduct this exercise both in the context of a “small” (single cohort) and a “large” insurance company. The option values are derived using two different approaches: 1) Nested Monte Carlo simulations; and 2) a discretization of the state space using a multidimensional grid and the solution of the one-period problems based on the corresponding Green’s function (similar to the approach introduced by Tanskanen and Lukkarinen (2003)). Our findings demonstrate the premium options generally have positive value and can place insurers at significant risk that must be managed even under the less restrictive information structure we propose here. This allows us to recognize the question posed in GS is important even though the specific contract structure they consider carries no risk to the insurer under the revised information structure.

The remainder of the paper is organized as follows. In Section 2, we briefly introduce the

¹The methodology applied by GS was originally proposed by Kling et al. (2006) in a different context. More precisely, Kling et al. calculate the value of paid-up premium options in a specific pension scheme using Monte Carlo techniques and find that the considered guarantees have positive value under several different assumptions about the flow of information. In particular, in contrast to GS, Kling et al. also calculate the (worst-case) value under the more realistic information structure we adopt here and find that it is positive. Nonetheless, of course our paper also has repercussions for their work and demonstrates that their results have to be interpreted with care.
contract as well as the valuation methodology applied in GS and demonstrate the insurer generally will be able to make a riskless profit even under the prescient policyholder behavior prescribed by GS. This result holds as long as the market is arbitrage-free and complete, and the life insurer is informed about its policyholders’ actions with regards to their policies. Hence, the value corresponding to the worst-case scenario from the insurer’s perspective identified by GS, which serves as an upper bound for the value calculated under the adjusted information structure, actually appears to be “too bad”. In Section 3, we value the options under the adjusted information structure and show that they carry no risk in the model proposed by GS. In Section 4, we consider the worst-case valuation of the premium payment options in alternative distribution rule frameworks.

In the main text we restrict ourselves to the presentation of the results as well as the intuition behind them. Formal proofs and derivations are provided in the Appendix.

2 The GS Model

Following GS, we assume a “complete, perfect, and frictionless market,” namely a Black-Scholes setup with one risky asset $S = (S(t))_{t \geq 0}$ with volatility $\sigma$ and a bank account in which funds are compounded continuously by the constant risk-free interest rate $r$.\(^2\) The basic endowment contract includes a constant death benefit $Y$ – determined by the Equivalence Principle under the guaranteed rate $g$ – as well as a variable survival benefit $V_T$ against annual, constant premium payments $B$.\(^3\) The “reserve” $V_t$ is defined based on $V_{t-1}$ via the following recursion relationship

$$V_t = \frac{1}{p_{x+t-1}} \cdot \left( (V_{t-1} + B) \cdot \left( 1 + \max \left\{ g, \alpha \left( \frac{S_t}{S_{t-1}} - 1 \right) \right\} \right) - q_{x+t-1} \cdot Y \right), \quad (1)$$

for $t = 1, 2, \ldots, T$, where $V_0 = 0$. The participation rate $\alpha$ and the guaranteed rate $g$ are fixed such that the value of the contract, that is the expected present value of benefits less premiums

\(^2\)Underlying the model is a complete, filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, P)$ on which the stochastic processes exist, and the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the “usual conditions”.

\(^3\)For the Equivalence Principle as well as for other concepts and/or notations from standard actuarial theory, we refer to Bowers et al. (1997).
under the risk-neutral measure (equivalent martingale measure) $Q$, is zero. Mortality evolves deterministically and actual survival rates within the portfolio of policyholders are assumed to be known.

In addition to this basic contract, GS introduce the *paid-up option*, which gives the insured the right to discontinue premium payments. If the option is exercised at time $t_0$, the death benefit is adjusted to $Y^{(t_0)}$ based on a single, under the guaranteed interest rate actuarially-fair benefit premium at time $t_0$ amounting to the current reserve, $V_{t_0}^{(t_0)} = V_{t_0}$. The survival benefit, $V_T^{(t_0)}$, is again calculated by Equation (1), where the benefit premiums $B$ are set to zero after time $t_0$.

Hence, the difference in discounted realized cash-flows after exercising the paid-up option at time $t_0$ will be (cf. Eq. (9) in GS)

$$C_{t_0}^{(t_0)}(Pa) = \sum_{k=t_0}^{T-1} p_{x+k} \cdot e^{-r(k+1-t_0)} \cdot \left( Y^{(t_0)} - Y \right)$$

$$+ T-t_0 p_{x+t_0} \cdot e^{-r(T-t_0)} \cdot \left( V_T^{(t_0)} - V_T \right) + \sum_{k=t_0}^{T-t_0} p_{x+t_0} \cdot e^{-r(k-t_0)} \cdot B,$$

where $C_T^{(T)}(Pa) = 0$.

The *resumption option* offers the possibility of resuming premium payments at time $t_1 > t_0$. In this case, again the death benefit is adjusted to $Y^{(t_0,t_1)}$ based on the current reserve as an up-front premium and additional annual premium payments $B$ using the Equivalence Principle with rate $g$. The survival benefit, $V_T^{(t_0,t_1)}$, is determined based on the current reserve level, $V_{t_1}^{(t_0,t_1)} = V_{t_1}^{(t_0)}$, and recursion (1). For more details, we refer to the original paper.

With the contract terms specified, GS need to make an assumption about policyholder behavior with respect to exercising these options. The authors make what they call a “behavioral-independent” assumption. I.e. they “do not base our [their] pricing on certain assumptions about particular policyholder exercise strategies” but instead “focus on the potential hazard of such options from the insurer’s viewpoint” by providing an “upper bound for any possible exercise scenario.” Specifically, they determine the option “value” as the expected value of the (pathwise)
maximal difference in discounted cash flows for a contract including and excluding the considered option under the risk-neutral measure, where the maximum is taken over all possible exercise dates. For example, for the paid-up option, they calculate (cf. Eq. (12) in GS).

\[ \Pi^{Opt} = E_Q^0 \left[ \max_{\tau \in \{1, 2, \ldots, T\}} \left\{ \tau p_x \cdot e^{-r \tau} \cdot C^{(\tau)}_\tau(Pa) \right\} \right]. \]  

One argument for applying this worst-case valuation method could be as follows: If policyholders exercise randomly, then – by pure chance – they could exercise optimally from an a posteriori perspective, i.e. as seen from the terminal time \( T \). Hence, policyholders could realize the “optimal” cash flow as seen from time \( T \), the risk-neutral expected discounted value of which is calculated in Equation (2). According to Kling et al. (2006), this analysis “quantifies the worst-case risk for the provider.”

While this reasoning may appear to be sound at first glance, standard complete-market arguments show this strategy does not present the worst-case scenario from the insurer’s point of view. Rather, this is given by the solution to the optimal stopping problem

\[ \tilde{\Pi}^{Opt} = \sup_{\tau \in \Upsilon_{\{0, 1, \ldots, T\}}} E_Q^0 \left[ \tau p_x \cdot e^{-r \tau} \cdot C^{(\tau)}_\tau(Pa) \right], \]  

where \( \Upsilon_{\{0, 1, \ldots, T\}} \) denotes all stopping times with values in \( \{0, 1, \ldots, T\} \). The optimal stopping rule can be determined in a backwards recursive scheme by solving the corresponding Bellman-Equations, i.e. by comparing exercise and continuation values according to the dynamic programming principle (see e.g. Bertsekas (1996) for an introduction to dynamic programming and optimal control).

To illustrate the problem with the worst-case valuation method expressed in Equation (2), let us fix an outcome \( \omega \) of the underlying sample space \( \Omega \) (i.e. a sample path of the stock price) and assume that the analysis from (2) reveals that some “prescient” exercise time \( t_0 \) is optimal from an a posteriori perspective. Suppose the optimal stopping rule from (3) does not yield exercising at \( t_0 \) to be optimal (i.e. the continuation value exceeds the exercise value), but the policyholder
still exercises. Then, disregarding mortality risk, the insurer will make a risk-free profit simply because the market cannot foresee the future. If the portfolio of policyholders is “infinitely large”, i.e. mortality risk diversifies, the exercise value will be sufficient to purchase a portfolio which perfectly replicates all future liabilities and the positive difference between the exercise and the continuation value can be taken as a risk-free profit. On the other hand, if the optimal stopping rule indicates that exercising at \( t_0 \) is optimal, but the \textit{a posteriori} optimal prescient time in (2) prescribes not to exercise and the insured chooses not to exercise, then the (smaller) continuation value will be sufficient to purchase a replicating portfolio and the positive difference between exercise and continuation value is a risk-free profit. Hence, the “super-replication” strategy resulting from (3) is minimax robust to any exercise strategy – even “lucky” or prescient ones. In particular, any exercise strategy that solves (3) is a worst-case scenario from the insurer’s point of view. Moreover, since any other exercise strategy is inferior, the corresponding value \( \tilde{\Pi}^{opt} \) is the least upper bound for any possible exercise behavior.

While the values \( \Pi^{opt} \) and \( \tilde{\Pi}^{opt} \) do not coincide in general, it is immediate to see that \( \Pi^{opt} \) is always greater than \( \tilde{\Pi}^{opt} \) and thus also presents an upper bound as correctly pointed out by GS:

\[
\Pi^{opt} = E^Q_0 \left[ \max_{\tau \in \{1, 2, \ldots, T\}} \left\{ \tau P_x \cdot e^{-r \tau} \cdot C^{(\tau)}_\tau(Pa) \right\} \right]
\]

\[
= \sup_{\tau(\omega) \in \{0, 1, \ldots, T\}, \tau \mathcal{F}_T \text{-measurable}} E^Q_0 \left[ \tau P_x \cdot e^{-r \tau} \cdot C^{(\tau)}_\tau(Pa) \right]
\]

\[
\geq \sup_{\tau \in \mathcal{Y} \{0, 1, \ldots, T\}} E^Q_0 \left[ \tau P_x \cdot e^{-r \tau} \cdot C^{(\tau)}_\tau(Pa) \right] = \tilde{\Pi}^{opt}.
\] (4)

Equation (4) shows that the value \( \Pi^{opt} \) identified by GS formally is only appropriate if the insurer does not learn about the policyholder’s decisions until the maturity of the contract \( T \), that is if – from the insurer’s perspective – \( \tau \) is \( \mathcal{F}_T \)-measurable. However, this assumption is unrealistic since the insurer is administering the contract and investing the underlying funds, and so will necessarily be aware of the cash flows in its accounts.\(^4\)

\(^4\)Solving the optimal control problem as in Equation (3) is also the more conventional approach for assessing exercise-dependent features in life insurance contracts in the academic literature (see Steffensen (2002), Bauer et al. (2010), and references therein).
The question of whether this upper bound nevertheless reflects “the potential hazard of such options from the insurer’s viewpoint” under the more coherent information structure where the insurer is aware of policy endorsements depends on how close the bound will be to the actual “worst-case value” $\tilde{\Pi}^{Opt}$ (“the value” in what follows). In particular, the adequacy of the numerical results and sensitivity analyses in the GS paper depend critically on how close the two quantities are. We analyze this relationship in the next section.

3 The Option Values under No Arbitrage

**Proposition 3.1.** All premium payment options in the GS model are worthless. Moreover, any exercise strategy is optimal.

To motivate this result, we first observe that for the basic contract, the participation-adjusted return on the reserve in some period $(k, k + 1]$ is independent of the evolution of the contract in $[0, k]$. In particular, this implies the time zero value of the basic contract can be written as

$$
\Pi_0 = B \left( e^{-rT} (1 + c)^T \sum_{j=0}^{T-1} (1 + c)^{-j} p_x - \sum_{j=0}^{T-1} e^{-rj} p_x \right) - Y \left( e^{-rT} (1 + c)^T \sum_{j=0}^{T-1} (1 + c)^{-(j+1)} q_x - \sum_{j=0}^{T-1} e^{-r(j+1)} q_x \right),
$$

where

$$
c = g + \alpha e^r BS(1, 1 + g/\alpha, 1) > g
$$

denotes the participation-adjusted expected rate and $BS(s_0, K, T)$ denotes the time zero Black-Scholes price for a European Call option with strike price $K$, expiration $T$, and time zero stock price $s_0$ (see Appendix A for the derivation of Equation (5)). Now, $g$ and $\alpha$ – and hence $c$ – are “calibrated” such that $\Pi_0 = 0$ (cf. Sec. 2). While it is immediate to see that $c = e^r - 1 = r^*$ is a solution to this problem, this solution can be shown to be unique given the definition of $Y$ (cf. Lemma B.2). As illustrated in Table 1, the numerical results obtained by GS are mostly in line
ON PREMIUM PAYMENT OPTIONS IN PARTICIPATING CONTRACTS

Table 1: Parameterizations from GS for different cases.

Therefore, the decision of whether a policyholder discontinues her premium payments or not does not affect the rate that current policy reserves or future premium payments will earn on average: The risk-neutral expected return equals the risk-free rate. Thus, the continuation value and the exercise value equal the current reserve and, in particular, coincide (see Equations (19) and (20) in the Appendix).\(^5\)

\(^5\)This implies that in this case there are no risk-free profits associated with exercising suboptimally since any exercise strategy is optimal.
4 Premium Options under Different Distribution Schemes

The key feature leading to the zero value result for the premium options in the GS setup is the independence of the annual participation-adjusted returns on the insured’s reserve account. In other words, the state of the insurance company with regards to its financial situation does not affect the distribution of returns. However, many alternative model specifications entail the – for many markets – more realistic assumption that the distribution rule is affected by the “state” of the company (see e.g. Grosen and Jørgensen (2000) or Miltersen and Persson (2003) for representative examples of distribution rules with state dependence).

To analyze the impact of such state-dependent distribution rules, as an example, we consider premium payment options under distribution rules similarly to those presented in Kling et al. (2007) and Bauer et al. (2006), or, more precisely, their “MUST” case. This distribution scheme describes obligatory payments to holders of participating policies as required in the German market for single premium term-fix contracts. For our purposes, it is necessary to modify the distribution scheme in order to consider periodic premium payments and death benefits. We do so in two settings. In Section 4.1, we consider the situation where policyholder decisions to stop or resume paying periodic premiums directly impact the financial condition of the insurer. We call this the “small” insurer case. In the next subsection, we assume a single policyholder’s decision to pay premiums is negligible in the context of a large pool of insureds and therefore does not impact the overall solvency position of the firm. We call this the “large” insurer case. Section 4.3 describes our numerical valuation approach and, in Section 4.4, we present results of both the “small” and “large” insurer cases.

4.1 Distribution in a “Small” Insurance Company

Consider a “small” insurance company that offers $T$-year participating endowment contracts to a (single) cohort of $l_x$ $x$-year old individuals against annual benefit premiums $B$. The (market value) balance sheet has three items: Assets $A_t$, Liabilities $L_t$, and a block of free reserves and equity $R_t = A_t - L_t$. From one policy anniversary to the next, the liabilities $L_t$, $t = 0, 1, 2, \ldots, T$, are
determined by the following equation (cf. Eq. (1)):

\[
\begin{cases}
    L_0 = 0, \\
    L_t = (L_{t-1} + B \cdot l_{x+t-1}) \cdot (1 + g) + \left[ \delta y (A_t^- - A_{t-1}^+) - g (L_{t-1} + B \cdot l_{x+t-1}) \right] + \\
    -(l_{x+t-1} - l_{x+t}) Y, \; t = 1, 2, \ldots, T,
\end{cases}
\]  

(6)

where \( Y \) is the constant death benefit determined by the Equivalence Principle at the guaranteed rate \( g \) and \( l_{x+t} = t p_x \times l_x \). That is, similarly to Bauer et al. (2006), the insureds’ account earns a guaranteed interest rate \( g \) as well as a fixed fraction \( \delta \) of the earnings on book values, which amount to a fraction \( y \) of the earnings on market values \( (A_t^- - A_{t-1}^+) \), \( t = 1, 2, \ldots, T \). Here, \( A_t^- \) denotes the company’s assets immediately before and \( A_t^+ \) denotes the company’s assets immediately after premium payments \( l_t \times B \), required contributions in case of a shortfall \( (L_t - A_t^-)^+ \), dividend payments \( d_t \) to the ownership for adopting the shortfall risk, and death benefit payments \( (l_{x+t-1} - l_{x+t}) Y \), i.e.

\[
A_t^+ = \max \{ A_t^- - d_t - (l_{x+t-1} - l_{x+t}) Y, L_t \} + l_{x+t} \times B, \; t = 1, 2, \ldots, T.
\]  

(7)

Moreover, as in Bauer et al. (2006), the remaining earnings on book values are being paid out as dividends, so that

\[
d_t = (1 - \delta) y (A_t^- - A_{t-1}^+) \cdot \mathbb{1} \left\{ \delta y (A_t^- - A_{t-1}^+) > g (L_{t-1} + B \cdot l_{x+t-1}) \right\} \\
+ \left[ y (A_t^- - A_{t-1}^+) - g (L_{t-1} + B \cdot l_{x+t-1}) \right] \times \mathbb{1} \left\{ \delta y (A_t^- - A_{t-1}^+) \leq g (L_{t-1} + B \cdot l_{x+t-1}) \leq y (A_t^- - A_{t-1}^+) \right\}.
\]  

(8)

For the evolution of the assets over each period – just like in Sections 2 and 3 – we rely on the basic Black-Scholes model, i.e.

\[
A_t^- = A_{t-1}^+ \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) + \sigma (W_t - W_{t-1}) \right\}, \; A_0^+ = E + l_x \times B, \; , t = 1, 2, \ldots, T,
\]

where \( W \) is a Brownian motion under the risk-neutral measure \( Q \), \( r \) is the risk-free interest rate, and \( E \) is the initial equity of the insurer. At maturity \( T \), the contract pays off \( L_T \) to the survivors,
i.e. each alive policyholder receives \( V_T = L_T/l_{x+T} \).

Thus, the contract value at time zero disregarding the premium payment options, i.e. the risk-neutral expected present value of the contract cash-flows, is (cf. Eq. (4) in GS)

\[
\Pi_0 = \sum_{j=0}^{T-1} e^{-r(j+1)} j(q_x Y + E_0^Q \left[ Tp_x e^{-rT} V_T \right] - \sum_{j=0}^{T-1} e^{-rj} jp_x B. \tag{9}
\]

As was pointed out in Bauer et al. (2006), some accounting freedom is necessary for offering these types of contracts, so that we assume that the insurer has some discretion in the choice of \( y \). Similar to before, in order to analyze the impact of the premium payment options, we assume that disregarding the premium options, the contract is “fair” in the sense of Doherty and Garven (1986), i.e. that \( y \) is chosen such that \( \Pi_0 = 0 \).

Consider now the paid-up (resumption) option, i.e. assume that the contract provides the policyholders with the possibility to collectively stop (resume) their premium payments at some policy anniversary \( t_0 \in \{1, 2, \ldots, T-1\} \) \( (t_1 \in \{t_0 + 1, 2, \ldots, T-1\}) \). For the paid-up option, just as before, the death benefit is adjusted to \( Y(t_0) \) and the survival benefit \( V_T^{(t_0)} \) is again calculated based on Equations (6), (7), and (8), where the premiums are set to zero after time \( t_0 \) and \( L_{t_0} = L_{t_0} \). Similarly, when exercising the resumption option at time \( t_1 > t_0 \), the death benefit \( Y(t_0,t_1) \) is adjusted to be actuarially fair under the guaranteed rate \( g \) and the survival benefit \( V_T^{(t_0,t_1)} \) is determined based on the concurrent liability level \( L_{t_1}^{(t_0,t_1)} = L_{t_1}^{(t_0)} \) and recursions (6), (7), and (8). Furthermore, we allow policyholders to – again – stop and resume premium payments in subsequent time periods, i.e. the options are not limited to exercising them once.

The value of the contract after exercising can then be determined in analogy with Equation (9). For example, the exercise value at time \( t_0 \) under the assumption that premium payments will not be resumed subsequently is of the form

\[
\Pi_{t_0}^{(t_0)} = \sum_{j=t_0}^{T-1} e^{-r(j+1-t_0)} j-t_0 \left[ Tp_x e^{-r(T-t_0)} V_T^{(t_0)} \right]. \tag{10}
\]

\(^{6}\)Note that since the policyholders are identical, an optimal decision for one policyholder will necessarily be optimal for all of them.
The \textit{worst-case} value as implied by arbitrage valuation theory can then be determined by solving the corresponding optimal control problem as explained in Section 2.

\subsection*{4.2 Distribution in a “Large” Company}

A potential problem of the application of the distribution rule outlined in the previous subsection for a “large” insurer offering a variety of products to a large number of heterogeneous customers is that the policyholder’s decision of whether or not to exercise influences the insurer’s funding situation. More specifically, the company’s solvency or reserve ratio $x_t = \frac{R_t}{L_t + B l_{x+t}}$, which is crucial to the return credited by the insurer, is directly affected by the amount of the premium payment. To illustrate this, we observe that with Equation (6),

$$ V_t = \frac{L_t}{l_{x+t}} = \frac{l_{x+t-1}}{l_{x+t}} \left[ \frac{(L_{t-1})}{l_{x+t-1}} + B \right] \left( 1 + \max \left\{ g, \delta \ y \left( \frac{A_t^- - A_t^+}{L_{t-1} + B l_{x+t-1}} \right) \right\} \right) - \frac{l_{x+t-1} - l_{x+t}}{l_{x+t-1}}$$

$$ = \frac{1}{p_{x+t-1}} \left[ (V_{t-1} + B) \left( 1 + \max \left\{ g, \delta \ y (1 + x_{t-1}) \left( \frac{A_{t-1}^-}{A_{t-1}^+} - 1 \right) \right\} \right) - q_{x+t-1} \right], \quad (11) $$

$t = 1, 2, \ldots, T$, which is of a very similar form to Equation (1), the key difference being the dependence on the solvency ratio. Now note, however, that

$$ x_t = \frac{A_t^+}{L_t + B l_{x+t}} - 1 = \max \left\{ A_t^+ - d_t - (l_{x+t-1} - l_{x+t}) Y - L_t, 0 \right\} $$

$$ (12) \quad \frac{L_t + B l_{x+t}}{L_t + B l_{x+t}} $$

decisively depends on the policyholder’s decision to pay premiums at time $t$. While this feature of an endogenous solvency ratio is natural in the “small” company case, it may yield unrealistic results when applied to a “large” company where the influence of the individual’s decision whether or not to exercise the premium option only very marginally affects the company’s solvency situation.

Whence, in order to separate individual decisions from the company’s evolution, we take the interpretation put forward in Kling et al. (2007) that if cash flows resulting from new business and premiums roughly compensate for death and surrender benefits at a company-wide scale, the evolution of a “term-fix” contract can be considered as an approximation for the evolution of the entire company. More precisely, for the evolution of the solvency ratio $x_t$, we take Equations (6),
(7), and (8) with $B = 0$ and $Y = 0$ (cf. Eqs. (1) et seq. in Bauer et al. (2006)), which yield

$$x_t = \max \left\{ 0, \frac{\left[ (1 + x_{t-1})(r_t + 1) - (1 - \delta) y(1 + x_{t-1}) r_t I\{\delta y(1 + x_{t-1}) > g\} \right]}{(1 + g) [\delta y (1 + x_{t-1}) r_t - g]} - 1 \right\},$$

where $r_t = \frac{A_t}{A_{t-1}} - 1$. In particular, we observe that then the evolution of the reserves $V_t$ solely depends on the realized return $r_t$ as well as on the past period states $(V_{t-1}, x_{t-1})'$.

The premium options are now defined analogously to the previous subsection with the same conventions for the definition of the values (see e.g. Eq. (9) and (10)). In particular, we notice that the death benefit after exercising will depend on the concurrent reserve level and – under the assumption that there is no subsequent exercise – remains constant for the remainder of the policy lifetime. Thus, the valuation problem is governed by the three-dimensional (discrete) Markov process $((J_t, V_t, x_t, Y_t)')_{t=0,1,...,T}$, where $J_t \in \{1, 2\}$ indicates whether premiums were paid the previous period ($J_t = 1$) or not ($J_t = 2$).

### 4.3 Implementation

In the case of the “small” company, we only consider contracts maturing after 2 and 3 periods that solely include a paid-up option. This is sufficient to obtain insights on the value and the exercise mechanism of these contracts, but it still allows us to use (nested) Monte Carlo simulations, where we make use of the observation that the valuation problem in the terminal period allows for a closed-form solution. More specifically, for the continuation value at time $t = T - 1$ (i.e. premium payments have not been terminated prior to or at time $T - 1$), since $\Pi_T = V_T$, we have:

$$\Pi^{\text{cont}}_{T-1} = e^{-r} q_{x+T-1} Y - B + E^Q_{T-1} \left[ e^{-r} p_{x+T-1} \Pi_T \right]$$

$$= (V_{T-1} + B) e^{-r} \left( (1 + g) + \delta y (1 + x_{T-1}) E^Q_{T-1} \left[ \frac{A_T}{A_{T-1}} - \left( 1 + \frac{g}{\delta y (1 + x_{T-1})} \right) \right] \right) - B \quad \text{(13)}$$

$$= (V_{T-1} + B) \left( e^{-r} (1 + g) + \delta y (1 + x_{T-1}) \text{BS}(1, 1 + \frac{g}{\delta y (1 + x_{T-1})}, 1) \right) - B,$$
where $x_{T-1}$ is defined as in Equation (12). Analogous formulas apply for the value $\tilde{\Pi}_{T-1}^{(\tau)}$ if the policyholder has exercised the paid-up option at time $\tau < T - 1$ and for $\Pi_{T-1}^{(T-1)}$, but the premium $B$ at time $T - 1$ is set to zero. In particular, the solvency ratio $x_{T-1}$ will depend on the premium payment. Then, we obtain for the contract value at time $T - 1$:

$$\tilde{\Pi}_{T-1} = \begin{cases} 
\tilde{\Pi}_{T-1}^{(\tau)}, & \tau < T - 1, \\
\max \left\{ \Pi_{T-1}^{(T-1)}, \Pi_{T-1}^{\text{cont}} \right\}, & \text{otherwise}.
\end{cases}$$

Similarly, we obtain $\tilde{\Pi}_{T-2}$, $\tilde{\Pi}_{T-3}$, \ldots, $\tilde{\Pi}_0$, where the one period problems of a form similar to (13) in each path are now carried out numerically using (nested) Monte Carlo simulations.

For our “large” company, similarly as for the surrender options analyzed in Tanskanen and Lukkarinen (2003) and Bauer et al. (2006), we implement a more sophisticated algorithm based on a discretization of the state space and the solution of the one-period problems based on the corresponding Green’s function. Intuitively, we solve the one-period problems in $[T - \tau, T - \tau + 1]$, $\tau = 1, 2, \ldots, T$, for each point of a grid in $\{1, 2\} \times \mathbb{R}^3$ at time $(T - \tau)$ based on the values at time $(T - \tau + 1)$, which are derived from the corresponding grid. More precisely, for every grid point at $(T - \tau)$, we approximate the value function at the end of the year as a function of the realized return $R_{T-\tau} = \frac{A_{T-\tau+1}}{A_{T-\tau}}$ via linear interpolation of $(M + 1)$ equidistant realizations with maximal value $R_{\text{MAX}}$, and then evaluate the corresponding integral. In order to determine the approximation based on the grid at $(T - \tau + 1)$, we use trilinear interpolation; trilinear interpolation not only presents a fast approach for evaluating the value functions for points off the grid but also is fairly accurate since it can be shown that the value function is piecewise linear in the reserve $V_{T-\tau+1}$ and the death benefit $Y_{T-\tau+1}$. Since the value function at time $T$ is $\Pi_T = V_T$, the value function at time zero can be determined by going backwards through the problem (we refer to Bauer et al. (2010) for a more detailed explanation of the underlying ideas).
4.4 Results

For the underlying parameters, we rely on the values used in Bauer et al. (2006), which correspond to regulatory requirements in Germany and/or values for a typical German life insurer. More precisely, we let the guaranteed interest rate \( g = 3.5\% \), the minimum participation rate \( \delta = 90\% \), the volatility parameter \( \sigma = 7.5\% \), and the (constant) risk-free interest rate \( r = 4\% \). Moreover, we consider \( x = 50 \) year-old policyholders paying an annual premium \( B = 1,000 \), where – for simplicity – mortality is assumed to evolve according to De Moivre’s Law with terminal age 100. As pointed out above, we then determine the accounting parameter \( y \) such that the value of the “basic” contract excluding premium options is approximately equal to zero.

“Small” Company

Here, we consider a population of \( l_{50} = 1,000 \) policyholders in a company endowed with initial capital \( E = 500,000 \). Then, in order to determine the accounting parameter leading to a “fair” basic contract, we calculate estimates for \( \Pi_0 \) based on Monte-Carlo simulations of \( 2.5 \cdot 10^6 \) paths for different values of \( y \) until \( \Pi_0 \) approximately equals to zero.

For \( T = 2 \), this procedure leads to a value of \( y = 0.3367 \). On this basis, we then determine the value of the contract including the paid-up option at \( t = 1 \) again based on \( 2.5 \cdot 10^6 \) realizations for the first year’s evolution. While with \( \tilde{\Pi}_0 = 3.67 \) the option does not appear too valuable, an analysis of the exercise probability is striking: it always is optimal to exercise. This appears to be due to the premium payment decision affecting the company’s solvency ratio \( x_1 \) as apparent from Equation (12), i.e. the decision to pay premiums leads to a dilution of the position of policyholders rendering the participation option less valuable. Since \( y \) is calibrated such that the value without any options is zero, paying premiums at time \( t = 1 \) is always suboptimal even for a very advantageous evolution over the first period.

This observation also remains valid for \( T = 3 \), i.e. it is always optimal to exercise at time \( t = 1 \). However, since here the fair \( y \) is slightly increased to \( y = 0.3568 \) due to lower expected returns in the future in the base case, the value of the option calculated based on \( 2,500^2 \) (nested) simulations
Figure 1: Value function at time zero $\Pi_0$ without the consideration of premium options as a function of the accounting parameter $y$ for the two numerical approaches.

is increased to $\tilde{\Pi}_0 = 9.92$. Hence, we find that at least in this situation, the proposition from GS that premium options may place the insurer at significant risk is ascertained.

“Large” Company

We consider contracts maturing in $T = 10$ years with paid-up as well as resumption options offered by a “large” company with initial solvency ratio $x_0 = 10\%$. To determine the fair accounting parameter $y$, similarly to the case of the “small” company, we now evaluate the “basic” contracts for different choices of $y$ based on both, Monte Carlo simulations and the discretization approach. Again, we use $2.5 \cdot 10^6$ paths for the estimation of the Monte Carlo values. For the discretization approach, we use a time-homogenous, equidistant grid for $((J_t, V_t, x_t, Y_t)')$ of size $2 \times (250 + 1) \times (500 + 1) \times (50 + 1)$ with minimal values $(1; 0; 0; 0)$ and maximal values $(2; 100,000; 2.5; 25,000)$. Furthermore, we set $M = 500$ and $R_{MAX} = 2.5$. The results are illustrated in Figure 1.

We notice that the difference between the value functions in both numerical approaches for the considered parameters is maximally 0.35 or 0.0035% of the sum of premiums, which falls in the
range of corresponding Monte Carlo errors for $2.5 \cdot 10^6$ simulations. Hence, on a reasonable scale, the deviations are hardly noticeable. We use $y = 0.39816$, which leads to a combined option value of approximately $\tilde{\Pi}_0 = 12.50$. While this result is considerably smaller than the comparable values in GS, this value may nevertheless be significant enough to place the insurer at significant risk if many policyholders exercise at the same time.

To further analyze the worst-case value, in Figure 2 the value functions $\Pi_0$ and $\tilde{\Pi}_0$ without and with the consideration of premium options for $y = 0.39816$ as a function of $x_0$, respectively, are displayed. We notice that the marginal worst-case value of the premium options, that is $\tilde{\Pi}_0 - \Pi_0$, is the greater the worse is the solvency situation of the insurance company as measured by $x_0$. This indicates that policyholders will be more inclined to exercise for relatively small solvency ratios. In particular, different policyholders could be inclined to exercise at the same time in situations where the company is in need for additional capital.

Hence, all in all, we are able to isolate two primary effects: On one hand, additional premium payments in a closed portfolio may dilute the ownership structure rendering the participation op-
tion less valuable; on the other hand, a meagerly capitalized company may be denied premium payments if these funds are mainly used for its re-capitalization rather than to the benefit of the policyholder. Especially this latter point documents the risk associated with offering these options as they present the most danger in already precarious situations.

5 Conclusions

In this paper, we reconsider the risks associated with premiums options often found in participating life insurance contracts as studied by Gatzert and Schmeiser (2008). In particular, we adopt a more realistic information structure that the insurance company knows about policy endorsements as they occur. Based upon this modification we find that their conclusion that “the substantial value of these options shows the necessity of appropriate pricing and adequate risk management” is inaccurate in their framework. Nevertheless, studying premium options of this type is an interesting problem, and the result that these options are worthless is an artifact of their contract specification. When analyzing these options in alternative settings, we find that they can carry a positive value and may place insurers at significant risk.

A caveat to this conclusion, however, may be the interpretation of participating schemes as an internal risk management solution for dealing with non-diversifiable risks such as policyholder behavior, since the insurer may have some discretion with respect to the bonus distribution in every period depending on the company’s state (see e.g. Norberg (2001)). More specifically, if – according to fundamental actuarial principles – the guaranteed (technical) calculation basis is chosen sufficiently conservative, participation schemes may be interpreted as a solution rather than a problem with respect to the impact of uncertain policyholder behavior.

References


Appendix

A Proof of Equation (5)

We will use the notation introduced in GS throughout this Appendix.

Lemma A.1. Let \( t \in \{0, 1, \ldots, T - 1\} \). For \( k = 0, 1, 2, \ldots, T - t \), we have

\[
V_{t+k} = \frac{1}{kp_{x+t}} \left[ V_t \sum_{j=0}^{k} (1+g)^j \alpha^{k-j} \sum_{t+1 \leq \mu_1 < \mu_2 \cdots < \mu_{k-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-j}} 
+ B \sum_{i=0}^{k-1} i p_{x+t} \sum_{j=0}^{k-i} (1+g)^j \alpha^{k-i-j} \sum_{t+i+1 \leq \mu_1 < \mu_2 \cdots < \mu_{k-i-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-i-j}} 
- Y \sum_{i=0}^{k-1} i q_{x+t} \sum_{j=0}^{k-i-1} (1+g)^j \alpha^{k-i-j-1} \sum_{t+i+2 \leq \mu_1 < \mu_2 \cdots < \mu_{k-i-j-1} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-i-j-1}} \right],
\]

where

\[
C_t = \left( \frac{S_t}{S_{t-1}} - (1+g/\alpha) \right)^+.
\]

Proof. First note that

\[
1 + \max \left\{ g, \alpha \left( \frac{S_t}{S_{t-1}} - 1 \right) \right\} = 1 + g + \alpha \max \left\{ 0, \frac{S_t}{S_{t-1}} - 1 - g/\alpha \right\} = (1+g) + \alpha C_t
\]

and we can proceed by induction in \( k \):

- For \( k = 0 \), we trivially have \( V_{t+0} = V_t \) and for \( k = 1 \), by Equation (3) in GS, we have

\[
V_{t+1} = \frac{1}{p_{x+t}} \left[ (V_t + B) \left( 1 + \max \left\{ g, \alpha \left( \frac{S_{t+1}}{S_t} - 1 \right) \right\} \right) - q_{x+t} Y \right]
= \frac{1}{p_{x+t}} [V_t ((1+g) + \alpha C_{t+1}) + B ((1+g) + \alpha C_{t+1}) - q_{x+t} Y].
\]
\[ V_{t+k+1} = \frac{1}{p_{x+t+k}} \left[ (V_{t+k} + B) \left( 1 + \max \left\{ g, \alpha \left( \frac{S_{t+k+1}}{S_{t+k}} - 1 \right) \right\} \right) - q_{x+t+k} Y \right] \]

Replacing \( V_{t+k} \) by the formula above according to the induction hypothesis, with some algebra, we obtain

\[
V_{t+k+1} = \frac{1}{k+1} \left[ V_t \sum_{j=0}^{k+1} (1 + g)^j \alpha^{k+1-j} \sum_{t+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-1-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-1-j}} 
+ B \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \sum_{t+i+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-1-i} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-1-i-j}} 
- Y \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \sum_{t+i+2 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-1-i-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-1-i-j}} \right],
\]

which completes the induction step and, hence, the proof.

\[ \square \]

**Corollary A.1.** Let \( t \in \{0, 1, \ldots, T-1\} \). For \( k = 0, 1, 2, \ldots, T-t \), we have

\[
V^{(t)}_{t+k} = \frac{1}{k+1} \left[ V_t \sum_{j=0}^{k} (1 + g)^j \sum_{t+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-j}} 
- Y^{(t)} \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \sum_{t+i+2 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-1-i-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-1-i-j}} \right].
\]

and for \( \nu \leq t \),

\[
V^{(\nu)}_{t+k} = \frac{1}{k+1} \left[ V_t^{(\nu)} \sum_{j=0}^{k} (1 + g)^j \sum_{t+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-j}} 
- Y^{(\nu)} \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \sum_{t+i+2 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-1-i-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-1-i-j}} \right].
\]

**Proof.** \( V_t^{(t)} = V_t \) (cf. p. 696 in GS) for the first equality and clearly \( V_{t+0}^{(\nu)} = V_t^{(\nu)} \) for the second equality. The induction step follows as in Lemma A.1 with \( B = 0 \) and \( Y = Y^{(t)} \) respectively \( Y = Y^{(\nu)} \).

\[ \square \]

\*Details are available from the authors upon request.*
Corollary A.2. Let \( t \in \{0, 1, \ldots, T - 1\} \). For \( k = 0, 1, 2, \ldots, T - t \) and \( \nu < t \), we have

\[
V_{t+k}^{(\nu,t)} = \frac{1}{kp_{x+t}} \left[ V_t^{(\nu)} \sum_{j=0}^{k} (1 + g)^j \alpha^{k-j} \sum_{t+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-j}} + B \sum_{i=0}^{k-1} \sum_{j=0}^{k-i} (1 + g)^j \alpha^{k-i-j} \sum_{t+i+1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-i-j} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-i-j}} \right]
\]

\[-Y^{(\nu,t)} \sum_{i=0}^{k-1} \sum_{j=0}^{k-i} (1 + g)^j \alpha^{k-i-j-1} \sum_{t+i+2 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-i-j-1} \leq t+k} C_{\mu_1} C_{\mu_2} \cdots C_{\mu_{k-i-j-1}} \].

Proof. \( V_t^{(\nu,t)} = V_t^{(\nu)} \) (cf. p. 697 in GS) and the induction step follows as in Lemma A.1 with \( Y = Y^{(\nu,t)} \). \( \square \)

Lemma A.2. For \( k = 0, 1, 2, \ldots, T \), we have

\[
E^Q \left[ e^{-r(T-k)} V_T 1_{\{T(x) > k\}} | \mathcal{F}_k, T(x) > k \right] = V_k \left( \frac{(1 + g)}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{-T-k} + B \sum_{j=0}^{k} \sum_{i=j}^{k} e^{-rj} \left( \frac{(1 + g)}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{-T-j} \]

where \( T(x) \) is the lifetime of an \( x \)-year old individual at time zero and \( BS(s_0, K, T) \) is the Black-Scholes price formula for a European Call option at time zero if the risk-free rate is \( r \), the underlying with volatility \( \sigma \) takes price \( s_0 \), the strike of the option is \( K \), and the expiry of the option is \( T \).

Proof. With Lemma A.1 we have

\[
E^Q \left[ e^{-r(T-k)} V_T 1_{\{T(x) > k\}} | \mathcal{F}_k, T(x) > k \right] = T-k p_{x+k} E^Q \left[ e^{-r(T-k)} V_T | \mathcal{F}_k \right] = T-k p_{x+k} E^Q \left[ e^{-r(T-k)} \frac{1}{T-k p_{x+k}} \left( V_k \sum_{j=0}^{k} \ldots + B \sum_{j=0}^{k} \ldots - Y \sum_{j=0}^{k} \ldots \right) | \mathcal{F}_k \right] = V_k e^{-r(T-k)} \sum_{j=0}^{k} (1 + g)^j \alpha^{T-k-j} \sum_{k+1 \leq \mu_1 < \cdots < \mu_{T-k-j} \leq T} E^Q [C_{\mu_1}] \ldots E^Q [C_{\mu_{T-k-j}}] + e^{-r(T-k)} B \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \alpha^{T-k-j-i} \sum_{k+i+1 \leq \mu_1 < \cdots < \mu_{T-k-i-j} \leq T} E^Q [C_{\mu_1}] \ldots E^Q [C_{\mu_{T-k-i-j}}] - Y e^{-r(T-k)} \sum_{i=0}^{k} \sum_{j=0}^{k-i} (1 + g)^j \alpha^{T-k-i-j-1} \sum_{k+i+2 \leq \mu_1 < \cdots < \mu_{T-k-i-j-1} \leq T} E^Q [C_{\mu_1}] \ldots E^Q [C_{\mu_{T-k-i-j-1}}],
\]

since \( C_{\mu_1}, C_{\mu_2}, \ldots \) are independent of \( \mathcal{F}_k \) and mutually independent. Moreover, they are identically distributed, so that

\[
E^Q [C_{\mu_1}] = E^Q [C_{\mu_2}] = \ldots
\]
Since there are \( \binom{n}{k} \) possibilities to choose \( k \) elements from \( n \) elements, we have

\[
E^Q \left[ e^{-r(T-k)} \right. 
\left. \mathbb{1}_{\{T(x) > T\}} \right| F_k, T(x) > k \]
\[
= V_k e^{-r(T-k)} \sum_{j=0}^{T-k} (1 + g)^j \alpha^{T-k-j} \left( \frac{T-k}{T-k-j} \right) E^Q [C_{\mu_1}]^{T-k-j} 
+ e^{-r(T-k)} B \sum_{i=0}^{T-k-1} \sum_{j=0}^{k-i} (1 + g)^i \alpha^{T-k-i-j} \left( \frac{T-k-i}{T-k-i-j} \right) E^Q [C_{\mu_1}]^{T-k-i-j} 
- Y e^{-r(T-k)} \sum_{i=0}^{T-k-1} \sum_{j=0}^{T-k-i-1} (1 + g)^{T-k-i-j} \left( \frac{T-k-i-1}{T-k-i-1-j} \right) E^Q [C_{\mu_1}]^{T-k-i-j-1} 
\]

by the binomial theorem. Now since

\[
E^Q [C_{\mu_1}] = E^Q \left[ e^{-r} \left( \frac{S_1}{S_0} - (1 + g/\alpha) \right)^+ \right] = BS(1, 1 + g/\alpha, 1),
\]

the claim follows. \( \square \)

**Corollary A.3.** For \( k = 0, 1, 2, \ldots, T, \nu < k \), we have

1. \( E^Q \left[ e^{-r(T-k)} \right. 
\left. \mathbb{1}_{\{T(x) > T\}} \right| F_k, T(x) > k \]
\[
= V_k \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k} 
- Y^{(k)} \sum_{j=0}^{T-k-1} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k-(j+1)}. 
\]

2. \( E^Q \left[ e^{-r(T-k)} \right. 
\left. \mathbb{1}_{\{T(x) > T\}} \right| F_k, T(x) > k \]
\[
= V^{(\nu)} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k} 
- Y^{(\nu)} \sum_{j=0}^{T-k-1} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k-(j+1)}. 
\]

3. \( E^Q \left[ e^{-r(T-k)} \right. 
\left. \mathbb{1}_{\{T(x) > T\}} \right| F_k, T(x) > k \]
\[
= V^{(\nu)} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k} 
+ B \sum_{j=0}^{T-k-1} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k-j} 
- Y^{(\nu,k)} \sum_{j=0}^{T-k-1} \left( \frac{1 + g}{e^r} + \alpha BS(1, 1 + g/\alpha, 1) \right)^{T-k-(j+1)}. 
\]
Similarly, is the value for a contract including a paid-up and a resumption option exercised at times

\[
\Pi_k := \sum_{j=0}^{T-k-1} Y e^{-r(j+1)} j_q x + k - \sum_{j=0}^{T-k-1} B e^{-rj} j_p x + k + E^Q \left[ e^{-r(T-k)} V_T 1_{(T(x) > T)} \right] F_k, T(x) > k
\]

\[
= V_k e^{-r(T-k)} (1 + c)^T - e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} B (1 + c)^{-j} j_p x + k - \sum_{j=0}^{T-k-1} B e^{-rj} j_p x + k
\]

\[-e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} Y (1 + c)^{-j} j_q x + k + \sum_{j=0}^{T-k-1} Y e^{-r(j+1)} j_q x + k
\]

(14)

is the value of the "basic" contract at time \( k \) and

\[
\Pi_k^{(k)} := \sum_{j=0}^{T-k-1} Y^{(k)} e^{-r(j+1)} j_q x + k + E^Q \left[ e^{-r(T-k)} V_T^{(k)} 1_{(T(x) > T)} \right] F_k, T(x) > k
\]

\[
= V_k e^{-r(T-k)} (1 + c)^T - e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} Y^{(k)} (1 + c)^{-j} j_q x + k
\]

\[+ \sum_{j=0}^{T-k-1} Y^{(k)} e^{-r(j+1)} j_q x + k
\]

(15)

is the value at time \( k \) for the contract solely including the paid-up option exercised at time \( k \), where

\[
c := g + \alpha e^r BS(1, 1 + g/\alpha, 1) > g.
\]

Similarly,

\[
\Pi_k^{(\nu)} := V_k^{(\nu)} e^{-r(T-k)} (1 + c)^T - e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} Y^{(\nu)} (1 + c)^{-j} j_q x + k
\]

\[+ \sum_{j=0}^{T-k-1} Y^{(\nu)} e^{-r(j+1)} j_q x + k
\]

(16)

is the value at time \( k \) for the contract solely including the paid-up option exercised at time \( \nu < k \), and

\[
\Pi_k^{(\nu,k)} := V_k^{(\nu)} e^{-r(T-k)} (1 + c)^T - e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} B (1 + c)^{-j} j_p x + k - \sum_{j=0}^{T-k-1} B e^{-rj} j_p x + k
\]

\[-e^{-r(T-k)} (1 + c)^T \sum_{j=0}^{T-k-1} Y^{(\nu,k)} (1 + c)^{-j} j_q x + k + \sum_{j=0}^{T-k-1} Y^{(\nu,k)} e^{-r(j+1)} j_q x + k
\]

(17)

is the value for a contract including a paid-up and a resumption option exercised at times \( \nu < k \) and \( k \), respectively.
B Proof of Proposition 3.1

Lemma B.1. The death benefit

\[ Y^B = Y^B(g) = B \frac{\sum_{j=0}^{T-1} p_x (1 + g)^{-j}}{\sum_{j=0}^{T-1} p_x g_{x+j} (1 + g)^{-(j+1)} + r p_x (1 + g)^{-T}} \]

is strictly increasing in \( g \).

Proof. From Lidstone (1905) we know that the level premium of an endowment insurance of 1,

\[ P = \frac{\sum_{j=0}^{T-1} p_x g_{x+j} (1 + g)^{-(j+1)} + r p_x (1 + g)^{-T}}{\sum_{j=0}^{T-1} p_x (1 + g)^{-j}} = \frac{1}{Y^1(g)}, \]

is strictly decreasing in \( g \), which is equivalent to the claim above. \( \square \)

Lemma B.2. Under the fairness condition for the basic contract, \( \Pi_0 = 0 \), we have

\[ c = e^r - 1 = r^* \]

Proof.

• Assume \( r^* > c \). Then by Lemma B.1, we have

\[ Y^1(r^*) = \frac{\hat{\mu}(r^*)}{\hat{A}^{(r^*)}_{\bar{x}:\bar{T}}} > Y^1(c) = \frac{\hat{\mu}(c)}{\hat{A}^{(c)}_{\bar{x}:\bar{T}}} \]

\[ \Rightarrow 0 > \hat{a}^{(c)}_{\bar{x}:\bar{T}} - \hat{a}^{(r^*)}_{\bar{x}:\bar{T}} > (1 + c)^T \left( \hat{a}^{(c)}_{\bar{x}:\bar{T}} - \hat{a}^{(r^*)}_{\bar{x}:\bar{T}} \right) \]

\[ \Rightarrow \frac{\hat{a}^{(r^*)}_{\bar{x}:\bar{T}}}{\hat{A}^{(r^*)}_{\bar{x}:\bar{T}}} > (1 + c)^T \left( \frac{\hat{a}^{(c)}_{\bar{x}:\bar{T}}}{\hat{A}^{(c)}_{\bar{x}:\bar{T}}} - \frac{\hat{a}^{(r^*)}_{\bar{x}:\bar{T}}}{\hat{A}^{(r^*)}_{\bar{x}:\bar{T}}} \right) \]

\[ \Rightarrow A^{(r^*)}_{\bar{x}:\bar{T}} > (1 + c)^T \left( \frac{\hat{a}^{(c)}_{\bar{x}:\bar{T}}}{\hat{A}^{(c)}_{\bar{x}:\bar{T}}} - \frac{\hat{a}^{(r^*)}_{\bar{x}:\bar{T}}}{\hat{A}^{(r^*)}_{\bar{x}:\bar{T}}} \right) \]

\[ \Rightarrow \frac{\hat{a}^{(r^*)}_{\bar{x}:\bar{T}}}{\hat{A}^{(r^*)}_{\bar{x}:\bar{T}}} > Y(r^*). \]  

\[ (18) \]

\[ ^8 \text{In order to keep our presentation concise, we will make use of classical actuarial notation, where } \cdot ^{(i)} \text{ denotes that the corresponding quantity is calculated at rate } i. \]
Now,

\[ 0 = \Pi_0 = \frac{(1 + c)^T}{(1 + r_T)^T} B \tilde{a}_{x:T}^{(c)} - B \tilde{a}_{x:T}^{(r^*)} - B Y^1(g) \left( \frac{(1 + c)^T}{(1 + r_T)^T} A_{x:T}^{(c)} - A_{x:T}^{(r^*)} \right) \]

\[ \Rightarrow Y^1(g) = \frac{\tilde{a}_{x:T}^{(r^*)} - \frac{(1 + c)^T}{(1 + r_T)^T} \tilde{a}_{x:T}^{(c)}}{A_{x:T}^{(r^*)} - \frac{(1 + c)^T}{(1 + r_T)^T} A_{x:T}^{(c)}}. \]

Thus, Equation (18) yields

\[ Y^1(g) > Y^1(r^*), \]

which by Lemma B.1 implies \( g > r^* > c > g \), i.e. a contradiction.

- Now assume \( c > r^* \). Then, again by Lemma B.1, \( Y^1(c) > Y^1(r^*) \). Analogously to the derivation of Equation (18), by interchanging \( c \) and \( r^* \), we obtain

\[ \frac{\tilde{a}_{x:T}^{(c)} - \frac{(1 + r_T)^T}{(1 + c)^T} \tilde{a}_{x:T}^{(r^*)}}{A_{x:T}^{(c)} - \frac{(1 + r_T)^T}{(1 + c)^T} A_{x:T}^{(r^*)}} > Y(c), \]

implying

\[ \frac{\tilde{a}_{x:T}^{(c)} - \frac{(1 + r_T)^T}{(1 + c)^T} \tilde{a}_{x:T}^{(r^*)}}{A_{x:T}^{(c)} - \frac{(1 + r_T)^T}{(1 + c)^T} A_{x:T}^{(r^*)}} = Y(g) > Y(c), \]

and hence \( g > c > g \), which again is a contradiction.

\[ \square \]

**Proof of Proposition 3.1.** Let us start by considering the exercise of the resumption option at time \( k > \nu > 0 \) when the paid-up option was exercised at time \( \nu \). With Lemma B.2, we have \( 1 + c = e^r \), so with Equations (16) and (17):

\[ \Pi_{k}^{(\nu)} = \Pi_{k}^{(\nu,k)} = V_{k}^{(\nu)}, \]  

(19)

so the continuation value equals the exercise value, and hence the option is worthless.

Similarly, regarding the exercise of the paid-up option at time \( k > 0 \), Equations (14) and (15) together with \( 1 + c = e^r \) yield

\[ \Pi_{k} = \Pi_{k}^{(k)} = V_{k}, \]  

(20)

so again the continuation value equals the exercise value, and hence the option is worthless.

\[ \square \]