# A Least-Squares Monte Carlo Approach to the Calculation of Capital Requirements*

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**Abstract**

The calculation of capital requirements for financial institutions entails a reevaluation of the company's assets and liabilities at some future point in time for a (large) number of stochastic forecasts of economic and firm-specific variables. The complexity of this nested valuation problem leads companies to struggle with the implementation.

Relying on a well-known method for pricing non-European derivatives, the current paper proposes and analyzes a novel approach to this computational problem based on least-squares regression and Monte Carlo simulations. We study convergence of the algorithm and analyze the resulting estimate for practically relevant risk measures. Importantly, we address the problem of how to choose the regressors (basis functions), and show that an optimal choice is given by the left singular functions of the corresponding valuation operator. Our numerical examples demonstrate that the algorithm can produce accurate results at relatively low computational costs, particularly when relying on the optimal basis functions.

*Keywords:* capital requirements, least-squares Monte Carlo, Value-at-Risk, singular value decomposition, guaranteed annuity option.

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# 1 Introduction

Many financial risk management applications entail a reevaluation of the company's assets and liabilities at some time horizon $\tau$ – sometimes called a *risk horizon* – for a large number of realizations of economic and firm-specific (state) variables. The resulting empirical loss distribution is then applied to derive risk measures such as Value-at-Risk (VaR) or Expected Shortfall (ES), which serve as the basis for capital requirements within several regulatory frameworks such as Basel III for banks and Solvency II for insurance companies. However, the high complexity of this *nested* computation structure leads firms to struggle with the implementation (Bauer et al., 2012).[1]

The present paper proposes an alternative approach based on least-squares regression and Monte Carlo simulations akin to the well-known Least-Squares Monte Carlo method (LSM) for pricing non-European derivatives introduced by Longstaff and Schwartz (2001). Analogously to the LSM pricing method, this approach relies on two approximations (Clément et al., 2002): On the one hand, the capital random variable, which can be represented as a risk-neutral conditional expected value at the risk horizon $\tau$, is replaced by a finite linear combination of functions of the state variables, so-called *basis functions*. As the second approximation, Monte Carlo simulations and least-squares regression are employed to estimate the coefficients in this linear combination. Hence, for each realization of the state variables, the resulting linear combination presents an approximate realization of the capital at $\tau$, and the resulting sample can be used for estimating relevant risk measures.

Although this approach is increasingly popular in practice for calculating economic capital particularly in the insurance industry (Barrie and Hibbert, 2011; Milliman, 2013; DAV, 2015) and has been used in several applied research contributions (Floryszczak et al., 2016; Pelsser and Schweizer, 2016, e.g.), these papers do not provide a detailed analysis of the properties of this algorithm or insights on how to choose the basis functions. Our work closes this gap in literature.

We begin our analysis by introducing our setting and the algorithm. As an important innovation, we frame the estimation problem via a *valuation operator* that maps future payoffs (as functionals of the state variables) to the conditional expected value at the risk horizon. In particular, we base our definition on a hybrid probability measure since simulations for risk estimation before the risk horizon are carried out under the physical measure whereas simulations for valuation after the risk horizon are carried out under a risk-neutral measure.

We formally establish convergence of the algorithm for the risk distribution (in probability) and for families of risk measures under general conditions when taking limits sequentially in the first and second approximation. In addition, by relying on results from Newey (1997) on the convergence of series estimators, we present conditions for the joint convergence of the two approximations in the general case and more explicit results for the practically relevant case of orthonormal polynomials.[2]

We then analyze in more detail the properties of the estimator for the important special case of VaR, which serves as the risk measure for regulatory frameworks such as Basel III or Solvency II. By building on ideas from Gordy and Juneja (2010), we show that for a fixed number of basis functions, the least-squares estimation of the regression approximation, while unbiased when viewed as an estimator for the individual loss, carries a positive bias term for this tail risk measure. It is important to note, however, that this result only pertains to the regression approximation but not the approximation of the actual loss variables via the linear combination of the basis functions – which is the crux of the algorithm. In particular, the adequacy of the estimate crucially depends

[1] As a consequence, many companies rely on approximations within so-called *standard models* or *standardized approaches*, which are usually not able to accurately reflect an company's risk situation and may lead to deficient outcomes (Liebwein, 2006; Pfeifer and Strassburger, 2008)

[2] We thank Giuseppe Benedetti for pointing us to this issue of joint convergence.

on the choice of basis functions.

This is where the operator formulation becomes especially useful. By expressing the valuation operator via its singular value decomposition (SVD), we show that under certain conditions, the (left) singular functions present an optimal choice for the basis functions. More precisely, we demonstrate that these singular functions approximate the valuation operator – and, thus, the distributions of relevant capital levels – in an optimal manner. The intuition is that similarly to an SVD for a matrix, the singular functions provide the most important dimensions in spanning the image space of the operator.

We comment on the joint convergence of the LSM algorithm under this choice and also the calculation of the singular functions. While in general the decomposition has to be carried out numerically, for certain classes of models it is possible to derive analytic expressions. As an important example class for applications, we discuss the calculation of the SVD – and, thus, the derivation of optimal basis functions – for models with Gaussian transition densities. In this case, it is straightforward to show that the underlying assumptions are satisfied. And, by following ideas from Khare and Zhou (2009), it is possible to derive the singular functions, which take the form of products of Hermite polynomials of linearly transformed states, by solving a related eigenvalue problem.

We illustrate our theoretical results considering two examples from life insurance in the context of annutization options. We first we consider a simple Guaranteed Annuity Option (GAO) within a pure endowment insurance contract in the Vasicek (1977) stochastic interest rate model. Following Boyle and Hardy (2003), we obtain a closed form solution for the valuation problem at the risk horizon so that we can conveniently compare the approximated realizations of the loss distribution with the exact ones. Our results demonstrate that the algorithm can produce accurate results at relatively low computational costs, although the interplay of the sample variance and the functional approximation is finical. We find that optimal basis functions improve the performance of the algorithm when compared to alternative basis functions with a different span.

As a second example, we consider popular annuitization guarantees within Variable Annuity contracts, so-called Guaranteed Minimum Income Benefits (GMIBs). In a setting with three stochastic risk factors (investment fund, interest, and mortality), we demonstrate that the algorithm still delivers reliable results when relying on sufficiently many basis functions and simulations. Here we emphasize that the optimal choice given by the singular functions not only determines the functional class – which are Hermite polynomials in this case, although of course different classes of univariate polynomials will generate the same span. But they also specify the most important combinations of stochastic factors, an indeed in our setting it turns out that higher-order combinations of certain risk factors are more important than lower-order combinations of others.[3] This latter aspect is very relevant in practical settings with high-dimensional state vectors, so that our results provide immediate guidance for these pressing problems.

### Related Literature and Organization of the Paper

Our approach is inspired by the LSM approach for derivative pricing and relies on corresponding results (Carriere, 1996; Tsitsiklis and Van Roy, 2001; Longstaff and Schwartz, 2001; Clément et al., 2002). A similar regression-based algorithm for risk estimation is independently studied in Broadie et al. (2015). Their results are similar to our sequential convergence results in Section 3.1, and the authors additionally introduce a *weighted* version of their regression algorithm. Moreover, Benedetti (2016) provides joint convergence results under an alternative set of conditions. However, these authors do not contemplate how to optimally choose the basis functions – although they emphasize the importance of this choice – which is a key contribution of our paper.

[3]We thank Baozhong Yang for pointing us in this direction.

As already indicated, the LSM approach enjoys popularity in the context of calculating risk capital for life insurance liabilities in practice and applied research, so that providing a theoretical foundation and guidance for its application are key motivating factors for this paper. A number of recent contributions discuss the so-called replicating portfolio approach as an alternative that enjoys certain advantages (Cambou and Filipović, 2016, e.g.), and Pelsser and Schweizer (2016) point out that the difference between the LSM versus the replicating portfolio calculation aligns with the so-called *regression-now* versus the so-called *regression-later* algorithm, respectively, for non-European option pricing (Glasserman and Yu, 2002). While a detailed comparison is beyond the scope of this paper, we note that although indeed in simple settings the performance of regress-later approaches appears superior (Beutner et al., 2013), the application comes with several caveats regarding the choice of the basis function and other complications in high-dimensional settings (Pelsser and Schweizer, 2016; Ha and Bauer, 2016).

The remainder of the paper is structured as follows: Section 2 lays out the simulation framework and the algorithm; Section 3 addresses convergence of the algorithm and analyzes the estimator in special cases; Section 4 discusses optimal basis functions and derives them in models with Gaussian transition densities; Section 5 provides the numerical examples; and, finally, Section 6 concludes. Proofs and technical details are relegated to the Appendix.

# 2 The LSM Approach

## 2.1 Simulation Framework

We assume that investors can trade continuously in a frictionless financial market with time finite horizon $T$ corresponding to the longest-term asset to liability of the company in view. Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space on which all relevant quantities exist, where $\mathbb{P}$ denotes the physical measure. We assume that all random variables in what follows are square-integrable (in $L^2(\Omega, \mathcal{F}, \mathbb{P})$). The sigma algebra $\mathcal{F}_t$ represents all information about the market up to time $t$, and the filtration $\mathbf{F}$ is assumed to satisfy the usual conditions.

The uncertainty with respect to the company's future assets and liabilities arises from the uncertain development of a number of influencing factors, such as equity returns, interest rates, demographic or loss indices, etc. We introduce the $d$-dimensional, sufficiently regular Markov process $Y = (Y_t)_{t\in[0,T]} = (Y_{t,1}, \dots, Y_{t,d})_{t\in[0,T]}$, $d \in \mathbb{N}$, the so-called *state process*, to model this uncertainty. We assume that all financial assets in the market can be expressed in terms of $Y$. Non-financial risk factors can also be incorporated (see e.g. Zhu and Bauer (2011) for a setting specific to life insurance that includes demographic risk). In this market, we take for granted the existence of a risk-neutral probability measure (martingale measure) $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which payment streams can be valued as expected discounted cash flows with respect to a given numéraire process $(N_t)_{t\in[0,T]}$.[4]

In financial risk management, we are now concerned with the company's financial situation at a certain (future) point in time $\tau$, $0 < \tau < T$, which we refer to as the *risk horizon*. More specifically, based on realizations of the state process $Y$ over the time period $[0, \tau]$ that are generated under the physical measure $\mathbb{P}$, we need to assess the *available capital* $C_\tau$, at time $\tau$ calculated as the market value of assets minus liabilities. This amount can serve as a buffer against risks and absorb financial losses. The *capital requirement* is then defined via a risk-measure $\rho$ applied to the capital random variable. For instance, if the capital requirement is cast based on VaR, the capitalization at time

[4] According to the *Fundamental Theorem of Asset Pricing*, this assumption is essentially equivalent to the absence of arbitrage. We refer to Schachermayer (2009) for details.

$\tau$ should be sufficient to cover the net liabilities at least with a probability $\alpha$, i.e. the additionally required capital is:

$$\mathrm{VaR}_\alpha(-C_\tau) = \inf\left\{ x \in \mathbb{R} \middle| \, \mathbb{P}\left(x + C_\tau \geq 0\right) \geq \alpha \right\}. \tag{1}$$

The capital at the risk horizon, for each realization of the state process $Y$, is derived from a market-consistent valuation approach. While the market value of traded instruments is usually readily available from the model ("mark-to-market"), the valuation of complex financial positions on the firm's asset side such as portfolios of derivatives and/or the valuation of complex liabilities such as insurance contracts containing embedded options typically requires numerical approaches. This is the main source of complexity associated with this task, since the valuation needs to be carried out for each realization of the process $Y$ at time $\tau$, i.e. we face a *nested* valuation problem.

Formally, the available capital is derived as a (risk-neutral) conditional expected value of discounted cash flows $X_t$, where for simplicity and to be closer to modeling practice, we assume that cash flows only occur at the discrete times $t = 1, 2, \ldots, T$ and that $\tau \in \{1, 2, \ldots, T\}$ :

$$C_\tau = \mathbb{E}^{\mathbb{Q}}\left[ \sum_{k=\tau}^{T} \frac{N_\tau}{N_k} X_k \middle| \left(Y_s\right)_{0 \leq s \leq \tau} \right]. \tag{2}$$

Note that within this formulation, interim asset and liability cash flows in $[0, \tau]$ may be aggregated in the $\sigma(Y_s, 0 \leq s \leq \tau)$-measurable position $X_\tau$. Moreover, in contrast to e.g. Gordy and Juneja (2010), we consider aggregate asset and liability cash flows at times $k \geq \tau$ rather than cash flows corresponding to individual asset and liability positions. Aside from notational simplicity, the reason for this formulation is that we particularly focus on situations where an independent evaluation of many different positions is not advisable or feasible as it is for instance the case within economic capital modeling in life insurance (Bauer et al., 2012).

In addition to current interest rates, security prices, etc., the value of the asset and liability positions may also depend on path-dependent quantities. For instance, Asian options depend on the average of a certain price index over a fixed time interval, lookback options depend on the running maximum, and liability values in insurance with profit sharing mechanisms depend on entries in the insurer's bookkeeping system. In what follows, we assume that – if necessary – the state process $Y$ is augmented so that it contains all quantities relevant for the evaluation of the available capital and still satisfies the Markov property (Whitt, 1986). Thus, we can write:

$$C_\tau = \mathbb{E}^{\mathbb{Q}}\left[ \sum_{k=\tau}^{T} \frac{N_\tau}{N_k} X_k \middle| Y_\tau \right].$$

We refer to the state process $Y$ as our *model framework*. Within this *framework*, the asset-liability projection *model* of the company is given by cash flow projections of the asset-liability positions, i.e. functionals $x_k$ that derive the cash flows $X_k$ based on the current state $Y_k$:[5]

$$\frac{N_\tau}{N_k} X_k = x_k\left(Y_k\right), \; \tau \leq k \leq T.$$

Hence, each *model* within our *model framework* can be identified with an element in a suitable function space, $\mathbf{x} = (x_\tau, x_{\tau+1}, ..., x_T)$ . More specifically, we can represent:

$$C_\tau\left(Y_\tau\right) = \sum_{j=\tau}^{T} \mathbb{E}^{\mathbb{Q}}\left[ x_j\left(Y_j\right) \middle| Y_\tau \right].$$

[5]Similarly to Section 8.1 in Glasserman (2004), without loss of generality, by possibly augmenting the state space or by changing the numéraire process (see Section 5), we assume that the discount factor can be expressed as a function of the state variables.

We now introduce the probability measure $\tilde{\mathbb{P}}$ via its Radon-Nikodym derivative:

$$\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}} = \frac{\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}.$$

**Lemma 2.1.** *We have:*

1. $\tilde{\mathbb{P}}(A) = \mathbb{P}(A),\ A \in \mathcal{F}_t,\ 0 \le t \le \tau.$
2. $\mathbb{E}^{\tilde{\mathbb{P}}}[X|\,\mathcal{F}_\tau] = \mathbb{E}^{\mathbb{Q}}[X|\,\mathcal{F}_\tau]$ *for every random variable* $X \in \mathcal{F}$.

Lemma 2.1 implies that we have:

$$C_\tau(Y_\tau) \;=\; \sum_{j=\tau}^{T}\mathbb{E}^{\tilde{\mathbb{P}}}[x_j(Y_j)|\,Y_\tau] = L\,\mathbf{x}(Y_\tau), \tag{3}$$

where the operator:

$$L : \mathcal{H} = \bigoplus_{j=\tau}^{T} L^2\left(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{P}}_{Y_j}\right) \to L^2\left(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau}\right) \tag{4}$$

is mapping a model to capital. We call $L$ in (4) the *valuation operator*. For our applications later in the text, it is important to note the following:

**Lemma 2.2.** *$L$ is a continuous linear operator.*

Moreover, for our results on the optimality of basis functions, we require compactness of the operator $L$. The following lemma provides a sufficient condition for $L$ to be compact in terms of the transition densities of the driving Markov process.

**Lemma 2.3.** *Assume there exists a joint density $\pi_{Y_\tau,Y_j}(y,x)$, $j = \tau, \tau+1, ..., T$, for $Y_\tau$ and $Y_j$. Moreover:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \pi_{Y_j|Y_\tau}(y|x)\,\pi_{Y_\tau|Y_j}(x|y)\,dy\,dx < \infty,$$

*where $\pi_{Y_j|Y_\tau}(y|x)$ and $\pi_{Y_\tau|Y_j}(x|y)$ denote the transition density and the reverse transition density, respectively. Then the operator $L$ is compact.*

The definition of $L$ implies that a model can be identified with an element of the Hilbert space $\mathcal{H}$ whereas (state-dependent) capital $C_\tau$ can be identified with an element of $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$. The task at hand is now to evaluate this element for a given model $\mathbf{x} = (x_\tau, \ldots, x_T)$ and to then determine the capital requirement via a (monetary) risk measure $\rho : L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau}) \to \mathbb{R}$ as $\rho(L\mathbf{x})$, although the *model* may change between applications as the exposures may change (e.g. from one year to the next or when evaluating capital allocations via the gradient of $\rho$ (Bauer and Zanjani, 2016, e.g.)).

One possibility to carry out this computational problem is to rely on *nested simulations*, i.e. to simulate a large number of scenarios for $Y_\tau$ under $\mathbb{P}$ and then, for each of these realizations, to determine the available capital using another simulation step under $\mathbb{Q}$. The resulting (empirical) distribution can then be employed to calculate risk measures (Lee, 1998; Gordy and Juneja, 2010). However, this approach is computationally burdensome and, for some relevant applications, may require a very large number of simulations to obtain results in a reliable range (Bauer et al., 2012). Hence, in the following, we propose and develop an alternative approach for such situations.

## 2.2 Least-Squares Monte-Carlo (LSM) Algorithm

As indicated in the previous section, the task at hand is to determine the distribution of $C_\tau$ given by Equation (3). Here, the conditional expectation causes the primary difficulty for developing a suitable Monte Carlo technique. This is akin to the pricing of Bermudan or American options, where "*the conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte-Carlo techniques*" (Clément et al., 2002). A solution to this problem was proposed by Carriere (1996), Tsitsiklis and Van Roy (2001), and Longstaff and Schwartz (2001), who use least-squares regression on a suitable finite set of functions in order to approximate the conditional expectation. In what follows, we exploit this analogy by transferring their ideas to our problem.

As pointed out by Clément et al. (2002), their approach consists of two different types of approximations. Proceeding analogously, as the first approximation, we replace the conditional expectation, $C_\tau$, by a finite combination of linearly independent basis functions $e_k(Y_\tau) \in L^2\left(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau}\right)$:

$$C_\tau \approx \widehat{C}_\tau^{(M)}(Y_\tau) = \sum_{k=1}^{M} \alpha_k \cdot e_k(Y_\tau). \tag{5}$$

We then determine approximate $\mathbb{P}$-realizations of $C_\tau$ using Monte Carlo simulations. We generate $N$ independent paths $(Y_t^{(1)})_{0\le t\le T}$, $(Y_t^{(2)})_{0\le t\le T}$,..., $(Y_t^{(N)})_{0\le t\le T}$, where we generate the Markovian increments under the physical measure for $t \in (0,\tau]$ and under the risk-neutral measure for $t \in (\tau, T]$.[6] Based on these paths, we calculate the realized cumulative discounted cash flows:

$$V_\tau^{(i)} = \sum_{j=\tau}^{T} x_j \left(Y_j^{(i)}\right),\ 1 \le i \le N.$$

We use these realizations in order to determine the coefficients $\alpha = (\alpha_1, \ldots, \alpha_M)$ in the approximation (5) by least-squares regression:

$$\hat{\alpha}^{(N)} = \operatorname{argmin}_{\alpha\in\mathbb{R}^M} \left\{ \sum_{i=1}^{N} \left[ V_\tau^{(i)} - \sum_{k=1}^{M} \alpha_k \cdot e_k\left(Y_\tau^{(i)}\right) \right]^2 \right\}.$$

Replacing $\alpha$ by $\hat{\alpha}^{(N)}$, we obtain the second approximation:

$$C_\tau \approx \widehat{C}_\tau^{(M)}(Y_\tau) \approx \widehat{C}_\tau^{(M,N)}(Y_\tau) = \sum_{k=1}^{M} \hat{\alpha}_k^{(N)} \cdot e_k(Y_\tau), \tag{6}$$

based on which we can then calculate $\rho\left(L\,\mathbf{x}\right) \approx \rho(\widehat{C}_\tau^{(M,N)})$.

In case the distribution of $Y_\tau$, $\mathbb{P}_{Y_\tau}$, is not directly accessible, we can calculate realizations of $\widehat{C}_\tau^{(M,N)}$ resorting to the previously generated paths $(Y_t^{(i)})_{0\le t\le T}$, $i = 1, \ldots, N$, or, more precisely, to the sub-paths for $t \in [0,\tau]$. Based on these realizations, we can determine the corresponding empirical distribution function and, consequently, an estimate for $\rho(\widehat{C}_\tau^{(M,N)})$. For the analysis of potential errors when approximating the risk measure based on the empirical distribution function, we refer to Weber (2007).

[6] Note that it is possible to allow for multiple *inner* simulations under the risk-neutral measure per *outer* simulation under $\mathbb{P}$ as in the algorithm proposed by Broadie et al. (2015). However, as shown in their paper, a single inner scenario as within our version will be the optimal choice when allocating a finite computational budget. The intuition is that the inner noise diversifies in the regression approach whereas additional outer scenarios add to the information regarding the relevant distribution.

# 3 Analysis of the Algorithm

## 3.1 Convergence

The following proposition establishes convergence of the algorithm described in Section 2.2 when taking limits sequentially:

**Proposition 3.1.** *$\widehat{C}_\tau^{(M)} \to C_\tau$ in $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$, $M \to \infty$, and $\widehat{C}_\tau^{(M,N)} \to \widehat{C}_\tau^{(M)}$, $N \to \infty$, $\tilde{\mathbb{P}}$-almost surely. Furthermore, $Z^{(N)} = \sqrt{N}\left[\widehat{C}_\tau^{(M)} - \widehat{C}_\tau^{(M,N)}\right] \longrightarrow$ Normal $(0, \xi^{(M)})$, where $\xi^{(M)}$ is provided in Equation (30) in the Appendix.*

We note that the proof of this convergence result is related to and simpler than the corresponding result for the Bermudan option pricing algorithm in Clément et al. (2002) since we do not have to take the recursive nature into account. The primary point of Proposition 3.1 is the convergence in probability – and, hence, in distribution – of $\widehat{C}_\tau^{(M,N)} \to C_\tau$ implying that the resulting distribution function of $\widehat{C}_\tau^{(M,N)}$ presents a valid approximation of the distribution of $C_\tau$ for large $M$ and $N$. The question of whether $\rho(\widehat{C}_\tau^{(M,N)})$ presents a valid approximation of $\rho(C_\tau)$ depends on the regularity of the risk measure. In general, we require continuity in $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$ as well as point-wise continuity with respect to almost sure convergence (see Kaina and Rüschendorf (2009) for a corresponding discussion in the context of convex risk measures). In the special case of orthogonal basis functions, we are able to present a more concrete result:

**Corollary 3.1.** *If $\{e_k,\, k = 1, \ldots, M\}$ are orthonormal, then $\widehat{C}_\tau^{(M,N)} \to C_\tau$, $N \to \infty$, $M \to \infty$ in $L^1(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$. In particular, if $\rho$ is a finite convex risk measure on $L^1(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$, we have $\rho(\widehat{C}_\tau^{(M,N)}) \to \rho\left(C_\tau\right)$, $N \to \infty$, $M \to \infty$.*

Thus, at least for certain classes of risk measures $\rho$, the algorithm produces a consistent estimate, i.e. if $N$ and $M$ are chosen *large enough*, $\rho(\hat{C}_\tau^{(M,N)})$ presents a viable approximation. In the next part, we make more precise what *large enough* means and, particularly, how large $N$ needs to be chosen relative to $M$.

## 3.2 Joint Convergence and Convergence Rate

The LSM algorithm approximates the capital level – which is given by the conditional expectation of the aggregated future cash flows $V_\tau = \sum_{j=1}^{T} x_j(Y_j^{(i)})$ – by its linear projection on the subspace spanned by the basis functions $e^{(M)}(Y_\tau) = (e_1(Y_\tau), \ldots, e_M(Y_\tau))'$ :

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[V_\tau | Y_\tau\right] \approx e^{(M)}(Y_\tau)'\, \hat{\alpha}^{(N)}.$$

Thus, the approximation takes the form of a *series estimator* for the conditional expectation. General conditions for the *joint* convergence of such estimators are provided in Newey (1997). Convergence of the risk measure then follows as in the previous subsection. We immediately obtain:[7]

**Proposition 3.2** (Newey (1997))**.** *Assume $Var(V_\tau|Y_\tau)$ is bounded and that for every $M$, there is a non-singular constant matrix $B$ such that for $\tilde{e}^{(M)} = B\, e^{(M)}$ we have:*

[7] Newey (1997) also provides conditions for uniform convergence and for asymptotic normality of series estimators. We refer to his paper for details.

- *The smallest eigenvalue of* $\mathbb{E}^{\mathbb{P}}\left[\tilde{e}^{(M)}(Y_\tau)\,\tilde{e}^{(M)}(Y_\tau)'\right]$ *is bounded away from zero uniformly in* $M$*; and*
- *there is a sequence of constants* $\xi_0(M)$ *satisfying* $\sup_{y\in\mathcal{Y}}\|\tilde{e}^{(M)}(y)\| \leq \xi_0(M)$ *and* $M = M(N)$ *such that* $\xi_0(M)^2\, M/N \to 0$ *as* $N \to \infty$*, where* $\mathcal{Y}$ *is the support of* $Y_\tau$*.*

*Moreover, assume there exist* $\psi > 0$ *and* $\alpha_M \in \mathbb{R}^M$ *such that* $\sup_{y\in\mathcal{Y}}|C_\tau(y) - e^{(M)}(y)'\,\alpha_M| = O(M^{-\psi})$ *as* $M \to \infty$*.*

*Then:*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(C_\tau - \hat{C}_\tau^{(M,N)}\right)^2\right] = O(M/N + M^{-2\,\psi}),$$

*i.e. we have joint convergence in* $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$*.*

In this result, we clearly see the influence of the two approximations: The functional approximation is reflected in the second part of the expression for the convergence rate. Here, it is worth noting that the speed $\psi$ will depend on the choice of the basis functions, emphasizing the importance of this aspect. The first part of the expression corresponds to the regression approximation, and in line with the second part of Proposition 3.1 it goes to zero linearly in $N$.

The result provides general conditions that can be checked for any selection of basis functions, although ascertaining them for each underlying stochastic model may be cumbersome. Newey also provides explicit conditions for the practically relevant case of power series. In our notation, they read as follows:

**Proposition 3.3** (Newey (1997))**.** *Assume* $Var(V_\tau|Y_\tau)$ *is bounded and that the basis functions* $e^{(M)}(Y_\tau)$ *consist of orthonormal polynomials, that* $\mathcal{Y}$ *is a Cartesian product of compact connected intervals, and that a sub-vector of* $Y_\tau$ *has a density that is bounded away from zero. Moreover, assume that* $C_\tau(y)$ *is continuously differentiable of order* $s$*.*

*Then, if* $M^3/N \to 0$*, we have:*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(C_\tau - \hat{C}_\tau^{(M,N)}\right)^2\right] = O(M/N + M^{-2s/d}),$$

*i.e. we have joint convergence in* $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$*.*

Hence, for orthonormal polynomials, the smoothness of the conditional expectation is important – which is not surprising given Jackson's inequality. First-order differentiability is required ($s \geq 1$), and if $s = 1$, the convergence of the functional approximation will only be of order $M^{-2/d}$, where $d$ is the dimension of the underlying model. Clearly, a more customized choice of the basis functions may improve on this rate.

We note that although $M/N$ enters the convergence rate, the conditions require $\xi_0(M)^2\, M/N \to 0$ in general and $M^3/N \to 0$ for orthonormal polynomials, effectively to control for the influence of estimation errors in the empirical covariance matrix of the regressors. Moreover, for common financial models the assumption of a bounded conditional variance or bounded support of the stochastic variables are not satisfied. Benedetti (2016) shows that if the distribution of the state process is known, convergence can still be ensured at a rate of $M^2 \log\{M\}/N \to 0$ under more modest – and in the financial context more appropriate – conditions. We refers to his paper for details.

Regarding the properties of the estimator beyond convergence, much rides on the first (functional) approximation that we discuss in more detail in the following Section 4. With regards to the second approximation, it is well-known that as the OLS estimate, $\widehat{C}_\tau^{(M,N)}$ is unbiased – though

not necessarily efficient – for $\widehat{C}_\tau^{(M)}$ under mild conditions (see e.g. Sec. 6 in Amemiya (1985)). However, this clearly does not imply that $\rho(\widehat{C}_\tau^{(M,N)})$ is unbiased for $\rho(\widehat{C}_\tau^{(M)})$. Proceeding similarly to Gordy and Juneja (2010) for the nested simulations estimator, in the next subsection we analyze this relationship in more detail for VaR.

### 3.3 LSM Estimate for Value-at-Risk

VaR is an important special case, since it is the risk measure applied in regulatory frameworks such as Basel III and Solvency II. VaR does not fall in the class of convex risk measures so that Corollary 3.1 does not apply. However, convergence immediately follows from Propositions 3.1-3.3:

**Corollary 3.2.** *We have:*

$$F_{\widehat{C}_\tau^{(M,N)}}(l) = \mathbb{P}(\widehat{C}_\tau^{(M,N)} \leq l) \to \mathbb{P}(C_\tau \leq l) = F_{C_\tau}(l),\ N \to \infty,\ M \to \infty,\ l \in \mathbb{R},$$

*and:*

$$F^{-1}_{\widehat{C}_\tau^{(M,N)}}(\alpha) \to F^{-1}_{C_\tau}(\alpha),\ N \to \infty, M \to \infty,$$

*for all continuity points $\alpha \in (0,1)$ of $F^{-1}_{C_\tau}$. Moreover, under the conditions of Propositions 3.2 and 3.3, we have joint convergence.*

Gordy and Juneja (2010) show that the nested simulations estimator for VaR carries a positive bias in the order of the number of simulations in the inner step. They derive their results by considering the joint density of the exact distribution of the capital at time $\tau$ and the error when relying on a finite number of inner simulations scaled by the square-root of the number of inner simulations. The following proposition establishes that their results carry over to our setting in view of the second approximation:

**Proposition 3.4** (Gordy and Juneja (2010))**.** *Let $g_N(\cdot,\cdot)$ denote the joint probability density function of $(-\hat{C}_\tau^{(M)}, Z^{(N)})$, and assume that it satisfies the regularity conditions from Gordy and Juneja (2010) collected in the Appendix. Then:*

$$\mathbb{E}\left[\widehat{VaR}_\alpha\left[-\widehat{C}_\tau^{(M,N)}\right]\right] = VaR_\alpha\left[-\widehat{C}_\tau^{(M)}\right] + \frac{\theta_\alpha}{N\bar{f}\left(VaR_\alpha\left[-\widehat{C}_\tau^{(M)}\right]\right)} + o_N(N^{-1}),$$

*where $\widehat{VaR}_\alpha\left[-\widehat{C}_\tau^{(M,N)}\right]$ denotes the $\lceil (1-\alpha)N \rceil$ order statistic of $\widehat{C}_\tau^{(M,N)}(Y_\tau^{(i)})$, $1 \leq i \leq N$ (the sample quantile), $\theta_\alpha = -\frac{1}{2}\frac{d}{d\mu}\left[\bar{f}(\mu)\mathbb{E}\left[\sigma^2_{Z^{(N)}}|-\widehat{C}_\tau^{(M)} = \mu\right]\right]_{\mu = VaR_\alpha\left[-\widehat{C}_\tau^{(M)}\right]}$, $\sigma^2_{Z^{(N)}} = \mathbb{E}\left[\left(Z^{(N)}\right)^2|Y_\tau\right]$, and $\bar{f}$ is the marginal density of $-\widehat{C}_\tau^{(M)}$.*

The key point of the proposition is that – similarly to the nested simulations estimator – the LSM estimator for VaR is biased. In particular, for large losses or a large value of $\alpha$, the derivative of the density in the tail is negative resulting in a positive bias. That is, ceteris paribus, on average the LSM estimator will err on the "conservative" side (see also Bauer et al. (2012)). However, note that this statement of course ignores the variance due to estimating the risk measure from the finite sample, which may well trump the inaccuracy due to bias – and unlike the nested simulations setting, here the two sources are governed by the same parameter $N$. Indeed, as is clear from Proposition 3.1, the convergence of the variance is of order $N$ and thus dominates the mean squared error for relatively large values of $N$ (the bias will enter as $O(N^{-2})$). Moreover, of course the result only pertains to the regression approximation but not the approximation of the capital variable via the linear combination of basis functions, which is at the core of the proposed algorithm.

# 4 Choice of Basis Functions

As demonstrated in Section 3.1, any set of independent functions will lead the LSM algorithm to converge. In fact, for the LSM method for pricing non-European derivatives, frequent choices of basis functions include Hermite polynomials, Legendre polynomials, Chebyshev polynomials, Fourier series, and even simple polynomials. While the choice is important for the pricing approximation (Glasserman, 2004, Sec. 8.6), several authors conclude based on numerical tests that the approach appears robust for typical problems when including a sufficiently large number of terms (see e.g. Moreno and Navas (2003) and also the original paper by Longstaff and Schwartz (2001)). A key difference between the LSM pricing method and the approach here, however, is that it is necessary to approximate the distribution over its entire domain rather than the expected value only. Furthermore, the state space for estimating a company's capital can be high-dimensional and considerably more complex than that of a derivative security. Therefore, the choice of basis functions is not only potentially more complex but also more crucial in the present context.

## 4.1 Optimal Basis Functions for a Model Framework

As illustrated in Section 2.1, we can identify the capital – as a function of the state vector at the risk horizon $Y_\tau$ – for a cash flow *model* $\mathbf{x}$ within a certain *model framework* $Y$ with the output of the linear operator $L$ applied to $\mathbf{x}$: $C_\tau(Y_\tau) = L\mathbf{x}(Y_\tau)$ (Eq. (3)). As discussed in Section 3.2, the LSM algorithm, in turn, approximates $C_\tau$ by its linear projection on the subspace spanned by the basis functions $e^{(M)}(Y_\tau)$, $P\,C_\tau(Y_\tau)$, where $P$ is the projection operator.

For simplicity, in what follows, we assume that the basis functions are orthonormal in $L^2(\mathbb{R}, \mathcal{B}, \mathbb{P}_{Y_\tau})$. Then we can represent $P$ as:

$$P\cdot = \sum_{k=1}^{M} \langle \cdot, e_k(Y_\tau)\rangle_{L^2(\mathbb{P}_{Y_\tau})}\, e_k.$$

Therefore, the LSM approximation can be represented via the *finite rank* operator $L_F = P\,L$, where we have:

$$\begin{aligned} L_F\mathbf{x} &= P\,L\,\mathbf{x} = \sum_{k=1}^{M} \langle L\mathbf{x}, e_k(Y_\tau)\rangle_{L^2(\mathbb{P}_{Y_\tau})}\, e_k \\ &= \sum_{k=1}^{M} \mathbb{E}^{\mathbb{P}}\left[ e_k(Y_\tau) \sum_{j=\tau}^{T} \mathbb{E}^{\tilde{\mathbb{P}}}\left[x_j(Y_j)|Y_\tau\right]\right] e_k = \sum_{k=1}^{M} \mathbb{E}^{\mathbb{P}}\left[ e_k(Y_\tau) \underbrace{\sum_{j=\tau}^{T} x_j(Y_j)}_{=V_\tau}\right] e_k \\ &= \sum_{k=1}^{M} \underbrace{\mathbb{E}^{\tilde{\mathbb{P}}}\left[e_k(Y_\tau)\,V_\tau\right]}_{\alpha_k} e_k, \end{aligned} \tag{7}$$

where the fourth equality follows by the tower property of conditional expectations.

It is important to note that under this representation, ignoring the uncertainty arising from the regression estimate, the operator $L_F$ gives the LSM approximation for *each* model $\mathbf{x}$ within the *model framework*. That is, the choice of the basis function *precedes* fixing a particular cash flow model (payoff). Thus, we can define *optimal basis functions* as a system that minimizes the distance between $L$ and $L_F$, so that the approximation is optimal with regards to all possible cash flow models within the framework:

**Definition 4.1.** *We call the set of basis functions* $\{e_1^*, e_2^*, ..., e_M^*\}$ *optimal in* $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$ *if:*

$$\{e_1^*, e_2^*, ..., e_M^*\} = \text{arginf}_{\{e_1,e_2,...,e_M\}} \|L - L_F\| = \text{arginf}_{\{e_1,e_2,...,e_M\}} \sup_{\|\mathbf{x}\|=1} \|L\mathbf{x} - L_F\mathbf{x}\|.$$

This notion of *optimality* has various advantages in the context of calculating risk capital. Unlike pricing a specific derivative security with a well-determined payoff, capital may need to be calculated for subportfolios or only certain lines of business for the purpose of capital allocation. Moreover, a company's portfolio will change from one calculation date to the next, so that the relevant cash flow model is in flux. The underlying model framework, on the other hand, is usually common to all subportfolios since the purpose of a capital framework is exactly the enterprise-wide determination of diversification opportunities and systematic risk factors. Also, it is typically not frequently revised. Hence, it is expedient here to connect the optimality of basis functions to the framework rather than a particular model (payoff).

## 4.2 Optimal Basis Functions for a Compact Valuation Operator

In order to derive optimal basis functions, it is sufficient to determine the finite-rank operator $L_F$ that presents the best approximation to the infinite-dimensional operator $L$. If $L$ is a compact operator, this approximation is immediately given by the *singular value decomposition* (SVD) of $L$ (for convenience, details on the SVD of a compact operator are collected in the Appendix). More precisely, we can then represent $L : \mathcal{H} \to L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$ as:

$$L\,\mathbf{x} = \sum_{k=1}^{\infty} \omega_k \,\langle \mathbf{x}, s_k \rangle\, \varphi_k, \tag{8}$$

where $\{\omega_k\}$ with $\omega_1 \geq \omega_2 \geq \ldots$ are the singular values of $L$, $\{s_k\}$ are the right singular functions of $L$, and $\{\varphi_k\}$ are the left singular functions of $L$ – which are exactly the eigenfunctions of $L\,L^*$. The following proposition demonstrates that the optimal basis functions are given by the left singular functions of $L$.

**Proposition 4.1.** *Assume the operator* $L$ *is compact. Then for each* $M$*, the left singular functions of* $L$ $\{\varphi_1, \varphi_2, \ldots, \varphi_M\} \in L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$ *are optimal basis functions in the sense of Definition 4.1. For a fixed cash flow model, we obtain* $\alpha_k = \omega_k \,\langle \mathbf{x}, s_k \rangle$.

The result that the left singular functions provide an optimal approximation may not be surprising given related results in finite dimensions. In particular, our proof is similar to the Eckart-Young-Mirsky Theorem on low-rank approximations of an arbitrary matrix. A sufficient condition for the compactness of the operator $L$ is provided in Lemma 2.3.

To appraise the impact of the two approximations simultaneously, we can analyze the joint convergence properties in $M$ and $N$ for the case of optimal basis functions. Here, in general, we have to check the conditions from Newey's convergence result (Prop. 3.2). We observe that the convergence rate associated with the first (functional) approximation depends on the parameter $\psi$, which in the present context derives from the speed of convergence of the singular value decomposition:

$$\begin{aligned} O(M^{-\psi}) = \inf_{\alpha_M} \sup_{y \in \mathcal{Y}} |C_\tau(y) - e^{(M)}(y)'\,\alpha_M| &\leq \sup_{y \in \mathcal{Y}} |\,L\,\mathbf{x}\,(y) - L_F\,\mathbf{x}\,(y)| \\ &= \sup_{y \in \mathcal{Y}} \left| \sum_{k=M+1}^{\infty} \omega_k \,\langle \mathbf{x}, s_k \rangle\, \varphi_k(y) \right|. \end{aligned} \tag{9}$$

In particular, we are able to provide an explicit result in the case of bounded singular functions.

**Proposition 4.2.** *Assume $Var(V_\tau|Y_\tau)$ is bounded and that the singular functions, $\{\varphi_k\}_{k=1}^{\infty}$, are uniformly bounded on the support of $Y_\tau$. Then, if $M^2/N \to 0$, we have:*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(C_\tau - \hat{C}_\tau^{(M,N)}\right)^2\right] = O(M/N + \omega_M^2),$$

*i.e. we have joint convergence in $L^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau})$.*

Comparing this convergence rate for singular functions to the general case from Proposition 3.2 and the orthonormal polynomial case from Proposition 3.3, we notice that the second term associated with the first (functional) approximation now is directly linked to the decay of the singular values. For integral operators, this rate depends on the smoothness of the kernel $k(x, y)$ (see Birman and Solomyak (1977) for a survey on the convergence of singular values of integral operators). In any case, Equation (9) that directly enters Newey's convergence result illustrates the intuition behind the optimality criterion: To choose a basis function that minimizes the distance between the operators for all $\mathbf{x}$, although in the Definition we consider the $L^2$-norm rather than the supremum.

The derivation of the SVD of the valuation operator of course depends on the specific model framework. In some cases, it is possible to carry out the calculations and derive analytical expressions for the singular values. In the next subsection, we determine the SVD – and, thus, optimal basis functions – in the practically highly relevant case of Gaussian transition densities. Here, the optimal basis functions correspond to Hermite polynomials of suitably transformed state variables and the singular values decay exponentially for $d = 1$ (Proposition 4.3), demonstrating the merits of this choice.

### 4.3 Optimal Basis Functions for Gaussian Transition Densities

In what follows, we consider a single cash flow at time $T$ only (generalizations follow analogously), and we assume that $(Y_t)$ is an $\mathbb{R}^d$-dimensional Markov process such that $(Y_\tau, Y_T)$ are jointly Gaussian distributed. We denote the $\tilde{\mathbb{P}}$-distribution of this random vector via:

$$\begin{pmatrix} Y_\tau \\ Y_T \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_\tau \\ \mu_T \end{pmatrix}, \quad \begin{pmatrix} \Sigma_\tau & \Gamma \\ \Gamma' & \Sigma_T \end{pmatrix}\right], \tag{10}$$

where $\mu_\tau$, $\mu_T$, $\Sigma_\tau$, and $\Sigma_T$ are the mean vectors and variance-covariance matrices of $Y_\tau$ and $Y_T$, respectively, and $\Gamma$ is the corresponding (auto) covariance matrix – which we assume to be non-singular.[8]

Denoting by $g(x; \mu, \Sigma)$ the normal probability density function at $x$ with mean vector $\mu$ and covariance matrix $\Sigma$, the marginal densities of $Y_\tau$ and $Y_T$ are $\pi_{Y_\tau}(x) = g(x; \mu_\tau, \Sigma_\tau)$ and $\pi_{Y_T}(y) = g(y; \mu_T, \Sigma_T)$, respectively. Mapping these assumption to the previous notation yields $\mathbf{x} = x_T$, $L : \mathcal{H} = L^2(\mathbb{R}^d, \mathcal{B}, \pi_{Y_T}) \to L^2(\mathbb{R}^d, \mathcal{B}, \pi_{Y_\tau})$, and:

$$C_\tau(Y_\tau) = L\mathbf{x}(Y_\tau) = \int_{\mathbb{R}^d} x_T(y)\, \pi_{Y_T|Y_\tau}(y|Y_\tau)\, dy,$$

where $\pi_{Y_T|Y_\tau}(y|x)$ denotes the transition density. In order to obtain optimal basis functions, the objective is to derive the SVD of $L$.

[8] The distribution in (10) is the unconditional distribution with known $Y_0$.

**Lemma 4.1.** *We have for the conditional distributions:*

$$Y_T|Y_\tau \sim N\left(\mu_{T|\tau}(x), \Sigma_{T|\tau}\right) \text{ and } Y_\tau|Y_T \sim N\left(\mu_{\tau|T}(y), \Sigma_{\tau|T}\right)$$

*with transition density and reverse transition density:*

$$\pi_{Y_T|Y_\tau}(y|x) = g(y; \mu_{T|\tau}(x), \Sigma_{T|\tau}) \text{ and } \pi_{Y_\tau|Y_T}(x|y) = g(x; \mu_{\tau|T}(y), \Sigma_{\tau|T}),$$

*respectively, where* $\mu_{T|\tau}(x) = \mu_T + \Gamma'\Sigma_\tau^{-1}(x-\mu_\tau)$, $\Sigma_{T|\tau} = \Sigma_T - \Gamma'\Sigma_\tau^{-1}\Gamma$, $\mu_{\tau|T}(y) = \mu_\tau + \Gamma\Sigma_T^{-1}(y-\mu_T)$, *and* $\Sigma_{\tau|T} = \Sigma_\tau - \Gamma\Sigma_T^{-1}\Gamma'$. *Moreover, L is compact in this setting.*

Per Proposition 4.1, the optimal basis functions are given by the left singular functions, which are in turn the eigenfunctions of $L\,L^*$. We obtain:

**Lemma 4.2.** *The operator* $LL^*$ *and* $L^*L$ *are integral operators:*

$$LL^*f(\cdot) = \int_{\mathbb{R}^d} K_A(\cdot, y)\, f(y)\, dy \text{ and } L^*Lf(\cdot) = \int_{\mathbb{R}^d} K_B(\cdot, x)\, f(x)\, dx,$$

*where the kernels are given by Gaussian densities:*

$$K_A(x, y) = g(y; \mu_A(x), \Sigma_A) \text{ and } K_B(y, x) = g(x; \mu_B(y), \Sigma_B)$$

*with*

- $\mu_A(x) = \mu_\tau + A(x - \mu_\tau)$, $A = \Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}$, *and* $\Sigma_A = \Sigma_\tau - A\Sigma_\tau A'$;
- $\mu_B(y) = \mu_T + B(y - \mu_T)$, $B = \Gamma'\Sigma_\tau^{-1}\Gamma\Sigma_T^{-1}$, *and* $\Sigma_B = \Sigma_T - B\Sigma_T B'$.

*We denote by* $\mathbb{E}_{K_A}[\cdot|x]$ *and* $\mathbb{E}_{K_B}[\cdot|y]$ *the expectation operators under the Gaussian densities* $K_A(x, \cdot)$ *and* $K_B(y, \cdot)$, *respectively.*

The problem of finding the singular values and the left singular functions therefore amounts to solving the eigen-equations:

$$\mathbb{E}_{K_A}\left[f(Y)|x\right] = \omega^2\, f(x).$$

We exploit analogies to the eigenvalue problem of the Markov operator of a first-order multivariate normal autoregressive (MAR(1)) process studied in Khare and Zhou (2009) to obtain the following:

**Lemma 4.3.** *Denote by* $P\Lambda P'$ *the eigenvalue decomposition of:*

$$\Sigma_\tau^{-1/2}\, A\, \Sigma_\tau^{1/2} = \Sigma_\tau^{-1/2}\, \Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1/2},$$

*where* $PP' = I$ *and* $\Lambda$ *is the diagonal matrix whose entries are the eigenvalues* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$ *of A. For* $y \in \mathbb{R}^d$, *define the transformation:*

$$z^P(y) = P'\Sigma_\tau^{-1/2}(y - \mu_\tau). \tag{11}$$

*Then for* $Y \sim K_A(x, \cdot)$, *we have:*

$$\mathbb{E}_{K_A}\left[z^P(Y)|x\right] = \Lambda\, z^P(x).$$

*Moreover,* $Var_{K_A}\left[z^P(Y)|x\right] = I - \Lambda^2$, $\mathbb{E}_{\pi_{Y_\tau}}\left[z^P(Y_\tau)\right] = 0$, *and* $Var_{\pi_{Y_\tau}}\left[z^P(Y_\tau)\right] = I$.

*Similarly, denote the diagonalization* $\Sigma_T^{-1/2}B\Sigma_T^{1/2} = Q\Lambda Q'$, *where* $Q'Q = I$ *and define the transformation:*

$$z^Q(x) = Q'\Sigma_T^{-1/2}(x - \mu_T). \tag{12}$$

*Then for* $X \sim K_B(y, \cdot)$, *we have:*

$$\mathbb{E}_{K_B}\left[z^Q(X)|y\right] = \Lambda\, z^Q(y),$$

$Var_{K_B}\left[z^Q(X)|y\right] = I - \Lambda^2$, $\mathbb{E}_{\pi_{Y_T}}\left[z^Q(Y_T)\right] = 0$, *and* $Var_{\pi_{Y_T}}\left[z^Q(Y_T)\right] = I$.

Therefore, for a random vector $Y|x$ in $\mathbb{R}^d$ that is distributed according to $K_A(x,\cdot)$, the components $z_i^P(Y)$ of $z^P(Y)$ are independently distributed with $z_i^P(Y) \sim N(\lambda_i\, z_i^P(x), 1-\lambda_i^2)$, where $z_i^P(x)$ is the $i$-th component of $z^P(x)$. Since eigenfunctions of standard Gaussian distributed random variables are given by Hermite polynomials, the SVD follows immediately from Lemma 4.3:

**Proposition 4.3.** *Denote the Hermite polynomial of degree $j$ by $h_j(x)$, that is:*[9]

$$h_0(x) = 1,\ h_1(x) = x,\ h_j(x) = \frac{1}{\sqrt{j}}\left(x\, h_{j-1}(x) - \sqrt{j-1} h_{j-2}(x)\right), \quad j = 2, 3, ...$$

*The singular values of $L$ in the current (Gaussian) setting are given by:*

$$\omega_m = \Pi_{i=1}^d \lambda_i^{k_i/2}, \quad m = (k_1, ..., k_d) \in \mathbb{N}_0^d, \tag{13}$$

*where $\mathbb{N}_0^d$ is the set of $d$-dimensional non-negative integers, and the corresponding right and left singular functions are:*

$$s_m(x) = \Pi_{i=1}^d h_{k_i}(z_i^Q(x)) \text{ and } \varphi_m(y) = \Pi_{i=1}^d h_{k_i}(z_i^P(y)),$$

*respectively.*

Combining the insights from Proposition 4.1 and Proposition 4.3, we immediately obtain:

**Corollary 4.1.** *Let $(m_k)_{k\in\mathbb{N}}$ be a reordering of $\{m\} = \{(k_1, ..., k_d) \in \mathbb{N}_0^d\}$ such that:*

$$\omega_{m_1} \geq \omega_{m_2} \geq \omega_{m_3} \geq \dots$$

*Then, in the current setting, optimal choices for the basis functions for the LSM algorithm in the sense of Definition 4.1 are given by:*

$$\varphi_k = \varphi_{m_k},\ k = 1, 2, 3, \dots$$

In the univariate case ($d = 1$), $A = \lambda_1$ is the square of the correlation coefficient between $Y_\tau$ and $Y_T$ – so that the singular values are simply powers of this correlation, decaying exponentially. Thus, the SVD takes the form:

$$L\,\mathbf{x}(Y_\tau) = \sum_{k=1}^{\infty} \left(\mathrm{Corr}(Y_\tau, Y_T)\right)^{k-1} \left\langle x_T, h_{k-1}\left(\frac{Y_T - \mu_T}{\Sigma_T}\right)\right\rangle_{\pi_{Y_T}} h_{k-1}\left(\frac{Y_\tau - \mu_\tau}{\Sigma_\tau}\right).$$

In particular, optimal basis functions are given by Hermite polynomials of the normalized Markov state – although other choices of polynomial bases will generate the same span so that the results will coincide.

In the general multivariate case, it is clear from Proposition 4.3 that the singular values of $L$ are directly related to eigenvalues of the matrix $A$ (or, equivalently, $B$), and there are $d$ vectors of indices $m$ such that $\sum_i k_i = 1$, $d^2$ vectors of indices such that $\sum_i k_i = 2$, etc. in Equation (13). The order of these singular values will determine the order of the singular functions in the SVD (8). In particular, after $\varphi_1(x) = 1$ with coefficient equaling $\langle x_T, 1\rangle = \mathbb{E}[x_T]$, the first nontrivial basis function is given by the singular function associated with the largest singular value – which according to (11) is a component of the linearly transformed normalized state vector. The

[9] See Kollo and Rosen (2006) for real- and vector-valued Hermite polynomials and the normalization employed here.

subsequent basis functions depend on the relative magnitudes of the different singular values. For instance, while for $1 > \lambda_1 > \lambda_2$ clearly $\sqrt{\lambda_1} > \sqrt{\lambda_1}^2 = \lambda_1$ and similarly for $\lambda_2$, it is not clear whether $\lambda_1 > \sqrt{\lambda_2}$ or vice versa – and this order will determine which combination of basis functions is optimal.

Thus, in the multi-dimensional case – and particularly in high-dimensional settings that are relevant for practical applications – is where the analysis here provides immediate guidance. Even if a user chooses the same function class (Hermite polynomials) or function classes with the same span (e.g., other polynomial families), it is unlikely that a naïve choice will pick the suitable combinations – and this choice becomes less trivial and more material as the number of dimensions increases.

From Proposition 3.1, we obtain sequential convergence. Joint convergence for (a class of) models $\mathbf{x}$ can be established by following Newey's approach from Propositions 3.2/3.3, or by relying on the results from Benedetti (2016) in case the parameters are known. While the Hermite polynomials do not satisfy the uniformly boundedness assumptions from Proposition 4.2, from Proposition 3.2 and the discussion following Proposition 4.2, it is clear that the convergence rate of the functional approximation is linked to the decay of the singular values ($O(\omega_M^2)$ in Prop. 4.2). In the current setting we have (Prop. 4.3):

$$\omega_M^2 = \omega_{m_M}^2 = \prod_{i=1}^{d} \lambda_i^{k_i} \leq \prod_{i=1}^{d} \max_{1\leq i\leq d}\{\lambda_i\}^{k_i} = \max_{1\leq i\leq d}\{\lambda_i\}^{\sum_i k_i},$$

where $\max_{1\leq i\leq d}\{\lambda_i\} < 1$ and there are $d$ vectors $m$ such that $\sum_i k_i = 1$, $d^2$ vectors $m$ such that $\sum_i k_i = 2$, etc. Thus, as in Proposition 3.3, the convergence is slowing down as the dimension $d$ of the state process increases, although the relationship here is exponential rather than polynomial.

In models with non-Gaussian transitions, while an analytical derivation may not be possible, we can rely on numerical methods to determine approximations of the optimal basis functions. For instance, Huang (2012) explains how to solve the associated integral equation by discretization methods, which allows to determine the singular functions numerically. Alternatively, Serdyukov et al. (2014) apply the truncated SVD to solve inverse problems numerically.

# 5 Applications

To illustrate the LSM algorithm and its properties, we consider two examples from life insurance: A Guaranteed Annuity Option (GAO) within a conventional pure endowment policy and a Guaranteed Minimum Income Benefit (GMIB) within a Variable Annuity contract. As indicated in the Introduction, the LSM algorithm is particularly relevant in insurance, especially in light of the new Solvency II regulation that came into effect in 2016. Here, the so-called *Solvency Capital Requirement* takes the form of a 99.5% VaR at the risk horizon $\tau = 1$.

## 5.1 Application to GAO

GAOs are common in many markets and, as described in Boyle and Hardy (2003), these options were a major factor in the demise of Equitable Life, the world's oldest life insurance company, in 2000. We consider the valuation of a GAO attached to a basic pure endowment policy under the Vasicek (1977) interest rate model. This framework has two advantages. First, following Boyle and Hardy (2003) and Pelsser (2003), it is possible to derive a closed form valuation formula. Hence, we can exactly simulate the capital level at the risk horizon and derive a closed form for the VaR. This allows us to appraise the performance of the LSM algorithm by comparing numerical results

to the "exact" quantities that are not subject to the functional approximation. Moreover, since the Vasicek model is driven by a simple Ornstein-Uhlenbeck (OU) process, it falls in the class of models considered in Section 4.3 and we can rely on the corresponding results to obtain optimal basis functions.

### 5.1.1 Payoff of the GAO and Valuation Formula

We consider a large portfolio of pure endowment policies with a GAO. In particular, we abstract from mortality risk (aggregate systematic risk as well as small sample risk), and to ease notation we derive all expressions for a single policyholder aged $x$ at time zero. Following standard actuarial notation we denote the $k$-year survival probability by ${}_kp_x$.

Under a plain pure endowment policy, a policyholder receives a fixed payment $P$ upon survival until the maturity date $T$ and nothing in case death occurs before time $T$. Thus, the time-$t$ value of the basic contract – if the policyholder is alive – is $P\,p(t,T)\,{}_{T-t}p_{x+t}$, where $p(t,T)$ is the value at time $t$ of a zero-coupon bond with maturity $T$. The benefit can be taken out as a fixed payment or can be converted into a life annuity under the concurrent market annuity payout rate, $m_{x+T}(T)$. In the latter case, policyholders will receive a payment of $P\,m_{x+T}(T)$ each year upon survival.

When the policy is equipped with a GAO, upon survival until $T$, the policyholder has the right to choose between (i) a fixed payment of $P$, (ii) a life annuity at the market rate $P\,m_{x+T}(T)$, or (iii) a life annuity with a guaranteed payout rate $g$ fixed at the policy's inception. Clearly, (i) and (ii) will result in the same (market) value, so that the time $T$ payoff for the pure endowment plus GAO is given by the maximum of options (ii) and (iii):[10]

$$P\,\max\{g, m_{x+T}(T)\}\,\underbrace{\sum_{k=1}^{\infty} {}_kp_{x+T}\,p(T,T+k)}_{=a_{x+T}(T)},$$

where $a_{x+T}(T)$ denotes the time $T$-value of an immediate annuity on an $(x+T)$-year old policyholder. We clearly have $m_{x+T}(T) = {}^1/_{a_{x+T}(T)}$, so that:

$$P\,\max\{g, m_{x+T}(T)\}\,a_{x+T}(T) = P + \underbrace{P\max\{g\,a_{x+T}(T) - 1, 0\}}_{=C(T)}.$$

Here, the bond prices within the annuity present value depend on the concurrent (time $T$) interest rate $r_T$, so that $C(T)$ takes the form of an interest rate derivative. For its valuation, we follow Vasicek (1977) and assume the interest rate evolves according to a unidimensional OU process:

$$dr_t = \alpha(\gamma - r_t)\,dt + \sigma\,dW_t, \tag{14}$$

under the physical measure $\mathbb{P}$, whereas the dynamics under the risk-neutral measure $\mathbb{Q}$ are given by:

$$dr_t = \alpha(\bar{\gamma} - r_t)\,dt + \sigma\,dZ_t. \tag{15}$$

Here $\alpha$ is the speed of mean reversion, $\gamma$ is the mean reversion level, $\sigma$ is the volatility, $\bar{\gamma} = \gamma - \lambda\sigma/\alpha$ where $\lambda$ is the market price of risk, and $(W_t)$ and $(Z_t)$ are standard Brownian motions under the physical measure and risk-neutral measure, respectively. Following Boyle and Hardy (2003), who

[10] Clearly, this entails the strong assumption on the policyholder's behavior to choose the value-maximizing option. While this may not be the case in a realistic setting with financial frictions, incomplete markets, or behavioral biases, we accept it here for illustrative purposes.

rely on the approach by Jamshidian (1989) for pricing options on a coupon bond, we obtain for the value of the GAO:

$$\begin{aligned} c(t) &= \mathbb{E}^{\mathbb{Q}}\left[{}_{T-t}p_{x+t}\, e^{-\int_t^T r_s\, ds}\, C(T)\,|r_t\right] && (16)\\ &= g\,{}_{T-t}p_{x+t}\sum_{k=1}^{\infty} {}_kp_{x+T}\,\left[p(t,T)\,\Phi(h) - K_k\,p(t,T+k)\,\Phi(h-\tilde{\sigma})\right]. && (17)\end{aligned}$$

Here $\Phi(\cdot)$ denotes the standard Normal cumulative distribution function,

$$\tilde{\sigma} = \sigma\sqrt{\frac{1-e^{-2\alpha(T-t)}}{2\alpha}}\;\frac{1-\exp(-\alpha\,k)}{\alpha}, \quad h = \frac{1}{\tilde{\sigma}}\log\left(\frac{p(t,T+k)}{p(t,T)K_k}\right) + \frac{\tilde{\sigma}}{2},$$

and the strike price $K_k$ is given by $p^*(T,T+k)$, where $r_T^*$ is the interest rate such that:

$$\sum_{k=1}^{\infty} {}_kp_{x+T}\,p^*(T,T+k) = 1/g,$$

and $p^*(T,T+k)$ is the price of a zero coupon bond priced at rate $r_T^*$. Thus, the price of the pure endowment plus GAO policy is:

$$v(t) = P\left(c(t) + p(t,T)\,{}_{T-t}p_{x+t}\right). \tag{18}$$

### 5.1.2 Capital Requirement for the GAO

The (available) capital at the risk horizon $\tau$ is given by the present value of assets $A_\tau$ minus liabilities $L_\tau$. For the single pure endowment plus GAO policy considered here, we obtain:

$$\begin{aligned} C_\tau = A_\tau - L_\tau &= A_\tau - \underbrace{P\,\left({}_{T-\tau}p_{x+\tau}\,p(\tau,T) + c(\tau)\right)}_{=v(\tau)}\\ &= A_\tau - P\,{}_{T-\tau}p_{x+\tau}\,p(\tau,T)\,\mathbb{E}^{\mathbb{Q}_T}\left[1 + C(T)|r_\tau\right], && (19)\end{aligned}$$

where $\mathbb{Q}_T$ denotes the $T$-forward measure, i.e. the risk-neutral measure when choosing $(p(t,T))$ as the numéraire process. For the dynamics of the risk-free rate, we have:

$$dr_t = \alpha(\bar{\gamma} - {}^{\sigma^2}/_{\alpha^2}\,(1-e^{-\alpha(T-t)}) - r_t)\,dt + \sigma\,dZ_t^T,$$

where $\left(Z_t^T\right)$ is a Brownian motion under $\mathbb{Q}_T$. The capital requirement can then be determined by a risk measure $\rho$ applied to $-C_\tau$: $\rho(-C_\tau)$ (see e.g. Eq. (1) in the case of VaR).

For simplicity, we ignore asset risk in what follows and set $A_\tau = 0$, so that we can express the capital requirement as $\rho\left(v(\tau)\right)$. Since the distribution of the risk-free rate under the physical measure is Normal, $r_\tau \sim N(\mu_\tau, \sigma_\tau^2)$ (see Lemma 5.1 below), and since $v(t)$ is decreasing in $r_t$, we can determine the capital in closed form for various risk measures. For instance, in the case of VaR, we obtain:

$$\mathrm{VaR}_\alpha = v(\tau, r_\tau = \mu_\tau - \Phi^{-1}(\alpha)\,\sigma_\tau). \tag{20}$$

For calculating the capital requirement via the LSM algorithm, we map the notation from the previous sections to the current setting. From Equation (19), it is clear that the relevant state process $Y_t = r_t$ is of dimension $d = 1$. Moreover, the cash flow functional $\mathbf{x} = x_T$, where:

$$x_T(r_T) = -v(T, r_T) = -P\,{}_{T-\tau}p_{x+\tau}\,p(\tau,T)\,[1 + C(T, r_T)]$$

and

$$C_\tau = L\mathbf{x}(r_\tau) = \mathbb{E}^{\mathbb{Q}_T}\left[x_T(r_T)|r_\tau\right].$$

To apply Proposition 4.3 to the current problem, we require the joint distribution of the state variables $r_\tau$ and $r_T$:

**Lemma 5.1.** *The joint distribution of $r_\tau$ and $r_T$ under $\tilde{\mathbb{P}}$ is:*

$$\begin{bmatrix} r_\tau \\ r_T \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_\tau \\ \mu_T \end{bmatrix}, \begin{bmatrix} \sigma_\tau^2, & e^{-\alpha(T-\tau)}\sigma_\tau^2 \\ e^{-\alpha(T-\tau)}\sigma_\tau^2, & \sigma_T^2 \end{bmatrix}\right),$$

*where we refer to the proof in the Appendix for explicit expressions of $\mu_\tau$, $\sigma_\tau$, etc. in terms of the parameters.*

From Proposition 4.3, we then have:

$$L\mathbf{x}(r_\tau) = \sum_{k=1}^{\infty} \rho^{k-1} \langle \mathbf{x}, h_{k-1} \rangle h_{k-1}(z(r_\tau)),$$

where $\rho = e^{-\alpha(T-\tau)}\sigma_\tau/\sigma_T$ and $z(r_\tau) = (r_\tau - \mu_\tau)/\sigma_\tau$. Importantly, since the first $n$ Hermite polynomials are spanned by other families of orthogonal polynomials and even simple monomials, other polynomial families will lead to equivalent results (ignoring possible numerical issues in the calculation of the regression coefficients). However, we can compare this family to other basis functions with a different functional form; following Proposition 3.1, we will have (sequential) convergence for any (square-integrable) choice of basis functions.

### 5.1.3 Numerical Results

We parametrize the model by using representative values. We set the initial interest rate $r_0 = 5\%$, and for the interest rate parameters we assume $\alpha = 15\%$ (speed of mean reversion), $\gamma = 5\%$ (mean reversion level), $\sigma = 1\%$ (interest rate volatility), $\lambda = 3\%$ (market price of risk), and $x = 55$ (age of the policyholder). For the mortality rates, for illustrative purposes, we use a simple De Moivre model with terminal age $\omega = 110$, so that ${}_kp_x = {}^{\omega-x-k}/_{\omega-x}$. For the insurance contract, we let the face value $P = 100$, the maturity $T = 10$, and the guaranteed annuity rate $g = {}^1/_9$. This rate corresponds to a (flat) interest rate of a little over 6%, so that the option will frequently be in-the-money. Finally, we set the risk horizon $\tau = 1$ as it is typical in insurance.

We start by analyzing the LSM approximation to the capital variable as we vary the number of basis functions. In Figure 1, we display the empirical density functions based on $N = 60,000$ Monte Carlo simulations for *exact* realizations according to Equation (17) and approximate realizations calculated via the LSM algorithm for different numbers of basis functions $M$. Here we rely on the optimal basis functions from Proposition 4.3 (Hermite polynomials). As is evident from the figure, the approximation becomes closer as $M$ increases, although already for low values of $M$ the LSM algorithm seems to capture the basic shape of the density. Hence, this first analysis seems encouraging that the LSM algorithm can provide viable results.

To appraise the influence of the choice of basis functions, in Figure 2 we compare the LSM approximation based on the singular functions as used in Figure 1 to a different choice of basis functions, namely the first $M$ elements of the Fourier basis. We observe that the approximation based on the (non-optimal) Fourier series is noticeably worse. In particular, from panel (2a) with $M = 4$, we find that the Fourier basis is not able to accurately reflect the shape of the density function. As the number of basis functions increases, of course the approximation becomes better

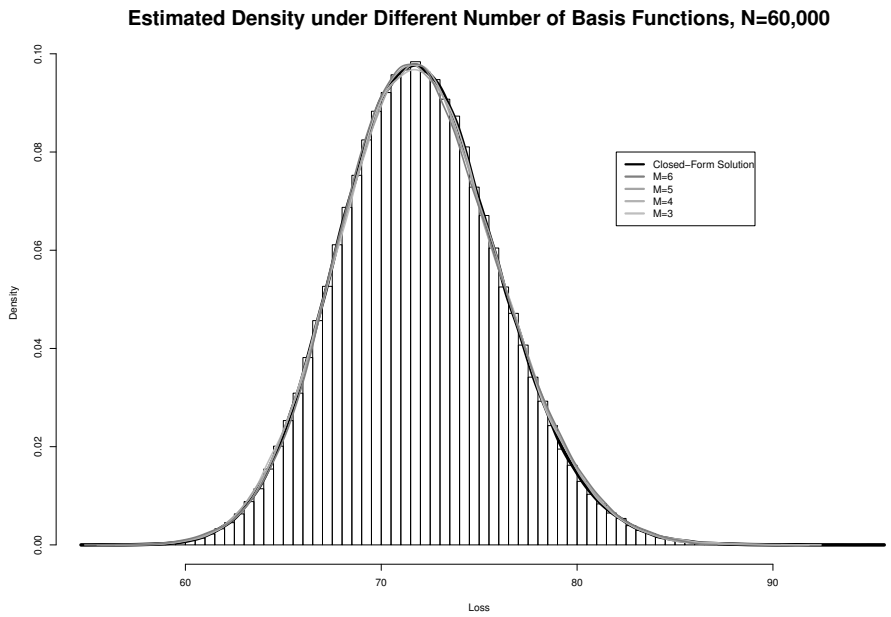
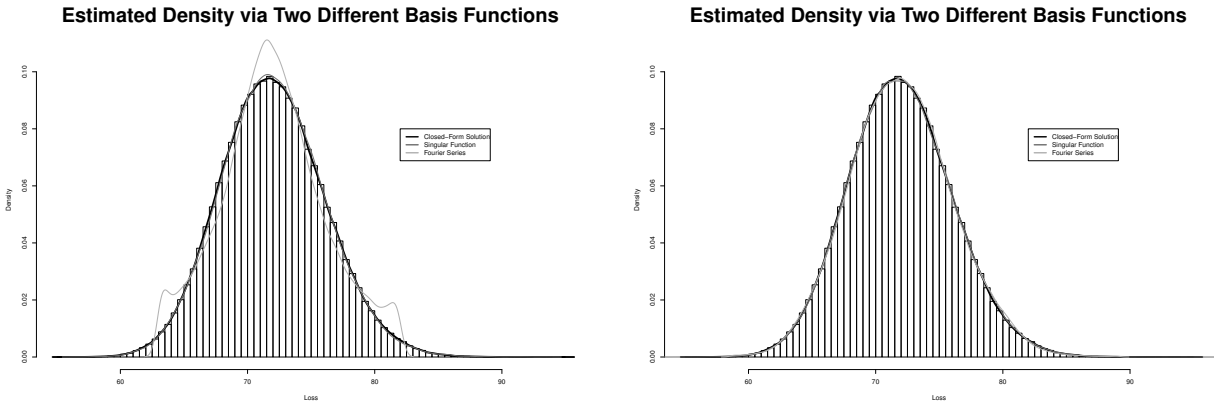



Figure 1: Empirical density functions of $v(\tau)$ based on $N = 60,000$ Monte Carlo realizations; exact and using the LSM algorithm with $M$ singular functions in the approximation.

as is evident from panel (2b) with $M = 10$. However, still the optimal basis functions provide a considerably better fit.

Table 1 reinforces this insight. Here, we show statistical differences between the empirical density functions based on $N = 700,000$ realizations (we report the mean of two-hundred runs) using, on the one hand, the exact realizations of the capital and, on the other hand, an LSM approximation. We compare differences for various choices of basis functions, both in view of the number of function terms $M$ and the function class (singular functions / polynomials vs. Fourier basis). For each combination, the table reports three common statistical distance measures: the Kolmogorov-Smirnov statistic (KS), the Kullback-Leibler divergence (KL), and the Jensen-Shannon divergence (JS). There are two key observations. First, the statistical distances are considerably smaller for the optimal choice of singular functions relative to the Fourier series. This holds for all combinations and distance measures, and, depending on the metric, the discrepancy is quite large. Second, the statistical difference increases for the singular functions as we add additional basis functions, i.e. as $M$ increases. The reason becomes clear when recalling our results on joint convergence: When increasing $M$, the error due to the regression approximation increases (the second approximation from Section 2.2 corresponding to the first term in the convergence order from Proposition 3.2). For the Fourier basis, on the other hand, adding a basis term decreases the distance. Generally, both aspects in the convergence rate are at work – as $M$ increases and with fixed $N$, the regression approximation worsens but the functional approximation improves – with either of them dominating in some cases.

The key application for the LSM algorithm in practice is calculating a company's capital requirement (economic capital), which is cast via a risk measure applied to the simulated distribution. Figure 3 shows results for the third quartile ($\text{VaR}_{75\%}$) and the 99.5% VaR ($\text{VaR}_{99.5\%}$). In both cases, we show results for different numbers of simulations $N$ used in the LSM algorithm on the $x$-axis. We use the first $M = 3$ (fixed) singular functions as basis functions. For each combination of risk measure and $N$, we run the LSM algorithm 300 times and determine the risk measure for




(a) $M = 4$, $N = 60,000$ (b) $M = 10$, $N = 500,000$

Figure 2: Empirical density functions of $v(\tau)$ based on $N$ Monte Carlo realizations; exact and using the LSM algorithm with different basis functions ($M$ terms)

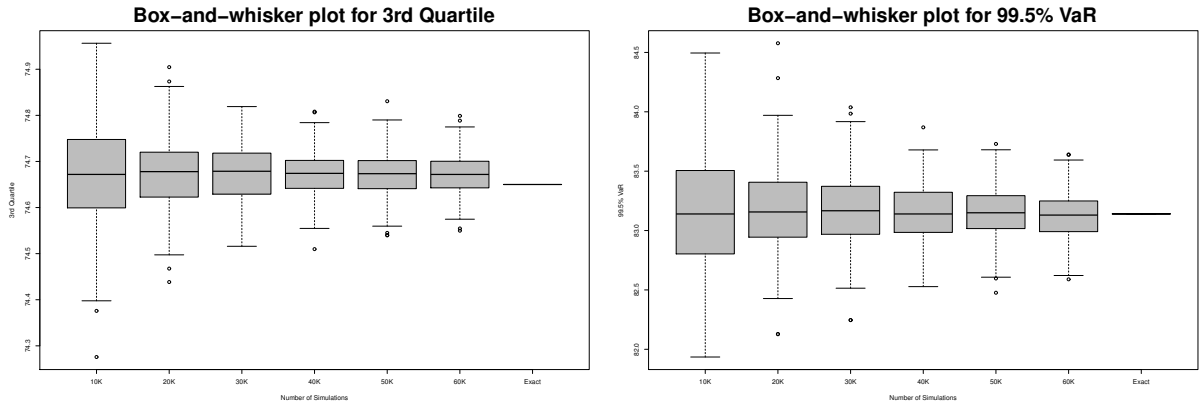
| Order | | Singular Functions | Fourier Series |
|---|---|---|---|
| | KS | $2.218 \times 10^{-3}$ | $6.601 \times 10^{-2}$ |
| $M = 3$ | KL | $1.413 \times 10^{-8}$ | $5.226 \times 10^{-5}$ |
| | JS | $5.465 \times 10^{-5}$ | $3.594 \times 10^{-3}$ |
| | KS | $2.291 \times 10^{-3}$ | $6.570 \times 10^{-2}$ |
| $M = 4$ | KL | $1.896 \times 10^{-8}$ | $9.582 \times 10^{-6}$ |
| | JS | $6.555 \times 10^{-5}$ | $1.507 \times 10^{-3}$ |
| | KS | $2.423 \times 10^{-3}$ | $6.208 \times 10^{-2}$ |
| $M = 5$ | KL | $2.421 \times 10^{-8}$ | $9.324 \times 10^{-6}$ |
| | JS | $7.435 \times 10^{-5}$ | $1.483 \times 10^{-3}$ |

Table 1: Statistical distances between the empirical density function based on the exact realizations and the LSM approximation using different basis functions; mean of two-hundred realizations of $N = 700,000$.

each run. Figure 3 provides box plots of the outcomes (the box presents the area between the first and third quartile, with the inner line placed at the median; the whisker line spans samples that are located closer than 150% of the interquartile range to the upper and lower quartiles, respectively (Tukey boxplot)).

The VaR formula from Equation (20) yields 74.65 and 83.14 for the third quartile and the 99.5% VaR, respectively. From Figure 3, it appears that the LSM algorithm produces viable results even with a relatively small number of simulations, e.g. ranging between about 74.5 to 74.9 for $\text{VaR}_{75\%}$ when using 20,000 simulations. However, this range becomes wider as we move towards the tail of the distribution, with the corresponding estimates for $\text{VaR}_{99.5\%}$ ranging between roughly 82.5 to 84. We observe a slight downward trend in the mean of the $\text{VaR}_{99.5\%}$ when increasing $N$ in line with the positive bias from Proposition 3.4. However, as also indicated in the discussion after the proposition, the bias is overshadowed by the sample variance resulting from the Monte Carlo estimation of the quantiles.

Increasing the number of simulations of course yields a more accurate estimation of the quantiles. In Figure 4a, we plot the distributions of 99.5% VaR for $N = 700,000$ and different choices for the




(a) Third quartile

(b) 99.5% VaR

Figure 3: Box-and-whisker diagrams for different risk measures (mean, third quartile, and 99.5% VaR) calculated using the LSM algorithm with different number of simulations $N$; the number of basis functions is fixed at $M = 3$.

number of basis functions $M$ (again box plots based on 300 runs). We find that the dispersion of the distribution becomes larger as the number of basis functions increases under the fixed number of simulations. Again, this emphasizes the importance of the joint convergence rate: When increasing $M$, to ascertain the approximation improves, it is necessary to simultaneously increase $N$.

The results are sensitive to changes in the parameters. For instance, in Figure 4b, we increase the volatility parameter ($\sigma$) from 1% to 2.5%. The VaR formula from Equation (20) yields 124.18 and we find that the range for $\text{VaR}_{99.5\%}$ at $\sigma = 2.5\%$ widens substantially relative to Figure 3. Thus, the required computational budget to obtain viable results may increase as the parameters change. Moreover, the positive bias arising from the VaR estimation is more evident in this case.

## 5.2 Application to GMIB

Within a Variable Annuity (VA) plus GMIB, at maturity $T$ the policyholder has the right to choose between a lump sum payment amounting to the current account value or a guaranteed annuity payment $b$ determined as a guaranteed rate applied to a guaranteed amount. GMIBs are popular riders for VA contracts: Between 2011 and 2013, roughly 15% of the more than $150 billion worth of Variable Annuities sold in the US contained a GMIB.[11] Importantly, GMIBs are subject to a variety of risk factors, including fund (investment) risk, mortality risk, and – as long term contracts – interest rate risk. Consequently, we consider its risk and valuation in a multivariate Markov setting for these three risk factors.

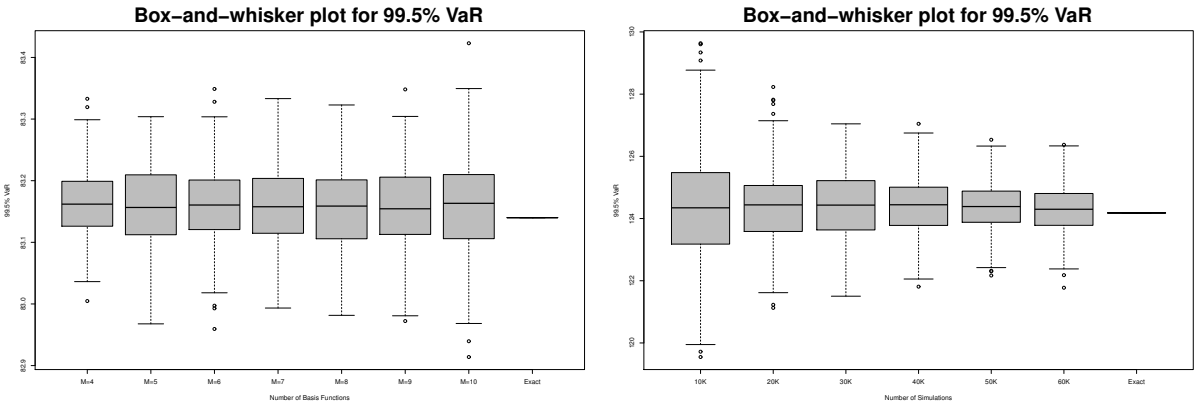
### 5.2.1 Model and Payoff of the GMIB

As in the previous section, we consider a large portfolio of GMIBs with policyholder age $x$, policy maturity $T$, and a fixed guaranteed amount – so that the guaranteed annuity payment $b$ is fixed at time zero.[12] The payoff of the VA plus GMIB at $T$ in case of survival is given by:

$$\max\left\{S_T,\ b\, a_{x+T}(T)\right\}, \tag{21}$$

[11] Source: Fact Sheets by the Life Insurance and Market Research Association (LIMRA).

[12] Some contract variants include path-dependent features such as ratchet guarantees.




(a) $N = 700,000$ (b) $M = 3$

Figure 4: Box-and-whisker diagrams for 99.5% VaR calculated using the LSM algorithm with different number of basis functions and $N$ fixed (a); and different number of simulations $N$ and $M$ fixed under an increased volatility parameter (b).

where $S_T$ is the underlying account value which evolves according to a reference asset net various fees (which we ignore for simplicity).

We consider a three-dimensional state process $Y_t$ governing financial and biometric risks:

$$Y_t = (q_t,\, r_t,\, \mu_{x+t})',$$

where $q_t$ denotes the log-price of the risky asset at time $t$, $r_t$ is the short rate, and $\mu_{x+t}$ is the force of mortality of an $(x+t)$-aged person at time $t$. We assume $Y_t$ satisfies the following stochastic differential equations under $\mathbb{P}$:

$$dq_t = \left(m - \frac{1}{2}\sigma_S^2\right)\, dt + \sigma_S\, dW_t^S, \tag{22}$$

$$dr_t = \alpha(\gamma - r_t)\, dt + \sigma_r\, dW_t^r, \tag{23}$$

$$d\mu_{x+t} = \kappa\mu_{x+t}\, dt + \psi\, dW_t^\mu, \tag{24}$$

where $m$ is the instantaneous rate of return of the risky asset, $\sigma_S$ is the asset volatility, $\kappa$ is an instantaneous rate of increment of mortality (Gompertz exponent), $\psi$ is the volatility of mortality, and $W_t^S$, $W_t^r$, and $W_t^\mu$ are standard Brownian motions under $\mathbb{P}$ with $dW_t^S\, dW_t^r = \rho_{12}\, dt$, $dW_t^S\, dW_t^\mu = \rho_{13} dt$, and $dW_t^r\, dW_t^\mu = \rho_{23} dt$. Note that the solutions to the above stochastic differential equations at time $t$ are Normal distributed so that we can derive the optimal basis function using the approach in Section 4.3.

The dynamics of $Y_t$ under the risk-neutral measure $\mathbb{Q}$ are given by:

$$dq_t = \left(r_t - \frac{1}{2}\sigma_S^2\right)\, dt + \sigma_S\, d\tilde{W}_t^S,$$

$$dr_t = \alpha(\bar{\gamma} - r_t)\, dt + \sigma_r\, d\tilde{W}_t^r,$$

$$d\mu_{x+t} = \kappa\mu_{x+t}\, dt + \psi\, d\tilde{W}_t^\mu,$$

where $\tilde{W}_t^S$, $\tilde{W}_t^r$, and $\tilde{W}_t^\mu$ are standard Brownian motions under $\mathbb{Q}$ with the same correlation coefficients. Here, for simplicity and without loss of generality, we assume that there is no risk premium

for mortality risk. Since the force of mortality is stochastic, the $k$-year survival probability ${}_kp_{x+t}$ is given by:

$$ {}_kp_{x+t} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+k}\mu_{x+s}\,ds}|Y_t\right], $$

and the time-$t$ value of the VA plus GMIB contract is:

$$ V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\mu_{x+s}\,ds}\max\left\{e^{q_T},\ b\,a_{x+T}(T)\right\}|Y_t\right]. \tag{25} $$

Since it is not possible to obtain an analytical expression for the GMIB, particularly when considering additional features such as step ups or ratchets, it is necessary to rely on numerical methods for valuation and estimating risk capital. To directly apply our LSM framework, we adjust the presentation by changing the numéraire to a pure endowment with maturity $T$ and maturity value one. The price of the VA plus GMIB at time $t$ is then:

$$ V(t) = {}_{T-t}E_{x+t}\,\mathbb{E}^{\mathbb{Q}_E}\left[\max\left\{e^{q_T},\ b\,a_{x+T}(T)\right\}|Y_t\right], \tag{26} $$

where $\tau \le t \le T$, ${}_{T-t}E_{x+t}$ is the price of the pure endowment contract at time $t$, and $\mathbb{Q}_E$ is the risk-neutral measure using the pure endowment contract as the numéraire.

Under our assumption, we obtain:

$$ {}_{T-t}E_{x+t} \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\mu_{x+s}\,ds}|Y_t\right] = A(t,T)\exp\left[-B_r(t,T)r_t - B_\mu(t,T)\mu_{x+t}\right] $$

since $(r_t)$ and $(\mu_t)$ are affine with

$$ B_r(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}, \quad B_\mu(t,T) = \frac{e^{\kappa(T-t)}-1}{\kappa}, $$

$$ \begin{aligned} A(t,T) = \exp\Bigg[\bar\gamma\left(B_r(t,T)-T+t\right) + \frac{1}{2}\Bigg\{ & \frac{\sigma_r^2}{\alpha^2}\left(T-t-2B_r(t,T)+\frac{1-e^{-2\alpha(T-t)}}{2\alpha}\right) \\ & + \frac{\psi^2}{\kappa^2}\left(T-t-2B_\mu(t,T)+\frac{e^{2\kappa(T-t)}-1}{2\kappa}\right) \\ & + \frac{2\rho_{23}\sigma_r\psi}{\alpha\kappa}\left(B_\mu(t,T)-T+t+B_r(t,T)-\frac{1-e^{-(\alpha-\kappa)(T-t)}}{\alpha-\kappa}\right)\Bigg\}\Bigg]. \end{aligned} $$

Thus, applying Itô's formula, the dynamics of the pure endowment price are:

$$ d_{T-t}E_{x+t} = {}_{T-t}E_{x+t}\left[(r_t+\mu_{x+t})dt - \sigma_r B_r(t,T)d\tilde W_t^r - \psi B_\mu(t,T)d\tilde W_t^\mu\right], $$

and from Brigo and Mercurio (2006), the new dynamics of $Y_t$ under $\mathbb{Q}_E$ for $\tau \le t \le T$ become:

$$ dq_t = \left(r_t - \frac{1}{2}\sigma_S^2 - \rho_{12}\sigma_S\sigma_r B_r(t,T) - \rho_{13}\sigma_S\psi B_\mu(t,T)\right)dt + \sigma_S dZ_t^S, \tag{27} $$

$$ dr_t = \left(\alpha(\bar\gamma - r_t) - \sigma_r^2 B_r(t,T) - \rho_{23}\sigma_r\psi B_\mu(t,T)\right)dt + \sigma_r dZ_t^r, \tag{28} $$

$$ d\mu_{x+t} = \left(\kappa\mu_{x+t} - \rho_{23}\sigma_r\psi B_r(t,T) - \psi^2 B_\mu(t,T)\right)dt + \psi\, dZ_t^\mu, \tag{29} $$

where $Z_t^S$, $Z_t^r$, and $Z_t^\mu$ are standard Brownian motions under $\mathbb{Q}_E$ with $dZ_t^S dZ_t^r = \rho_{12}dt$, $dZ_t^S dZ_t^\mu = \rho_{13}dt$, and $dZ_t^r dZ_t^\mu = \rho_{23}dt$.

Again proceeding similarly to the previous section, we ignore unsystematic mortality risk and the asset side in the calculation of the risk capital for the VA plus GMIB contract, and estimate the risk measure $\rho(V(\tau, Y_\tau))$ via the LSM algorithm. In particular, the cash flow functional in the current setting is $\mathbf{x} = x_T$ with:

$$x_T(Y_T) = -V(T) = -\max\{e^{q_T}, ba_{x+T}(T)\}$$

and

$$C_\tau = L\,\mathbf{x}(Y_\tau) = {}_{T-\tau}E_{x+\tau}\,\mathbb{E}^{\mathbb{Q}_E}\left[x_T(Y_T)|Y_\tau\right].$$

To apply our results on optimal basis functions, we require the joint distribution of $Y_\tau$ and $Y_T$:

**Lemma 5.2.** *From (22)–(24) and (27)–(29), the joint (unconditional) distribution of $Y_\tau$ and $Y_T$ under $\tilde{\mathbb{P}}$ is:*

$$\begin{pmatrix} Y_\tau \\ Y_T \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_\tau \\ \mu_T \end{pmatrix}, \quad \begin{pmatrix} \Sigma_\tau & \Gamma \\ \Gamma' & \Sigma_T \end{pmatrix}\right],$$

*where we refer to the proof in the Appendix for explicit expressions of $\mu_\tau$, $\mu_T$, $\Sigma_\tau$ etc. in terms of the parameters.*

Thus we can apply the results from Proposition 4.3 to derive optimal basis functions. More precisely, for any non negative integer vector $l = (l_1, l_2, l_3)$, $\omega_{|l|} = \lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}$ is a squared singular value of $L$, and the corresponding left singular functions is:

$$\varphi_l(x) = h_{l_1}(z_1^P(x))h_{l_2}(z_2^P(x))h_{l_3}(z_3^P(x)).$$

Thus, in order to find the set of optimal basis functions for the LSM algorithm consisting of $M$ functions, we need to calculate $\omega_{|m|}$ for $m = (m_1, m_2, m_3)$ such that $|m| \leq M$, order them, and then determine the associated functions.

### 5.2.2 Numerical Results

As in the previous application, we set the model parameters using representative values. The initial price of the risky asset is one hundred – so $q_0 = 4.605$ – and for the risky asset parameters we assume $m = 5\%$ (instantaneous rate of return) and $\sigma_S = 18\%$ (asset volatility). The initial interest rate is assumed to be $r_0 = 2.5\%$, $\alpha = 25\%$ (speed of mean reversion), $\gamma = 2\%$ (mean reversion level), $\sigma_r = 1\%$ (interest rate volatility), and $\lambda = 2\%$ (market price of risk). For the mortality rate, we set $x = 55$ (age of the policyholder), $\mu_{55} = 1\%$ (initial value of mortality), $\kappa = 7\%$ (instantaneous rate of increment), and $\psi = 0.12\%$ (mortality volatility). For correlations, we assume $\rho_{12} = -30\%$ (correlation between asset and interest rate), $\rho_{13} = 6\%$ (correlation between asset and mortality rate), and $\rho_{23} = -4\%$ (correlation between interest rate and mortality rate). For the insurance contract, we let the maturity $T = 15$, and the guaranteed annuity payout $b = 10.83$ per year.[13] We set the risk horizon $\tau = 1$ as in the previous application.

With the above parameters, the eigenvalues of $A$ are $\lambda_1 = 0.14898$, $\lambda_2 = 0.06712$, and $\lambda_3 = 0.00035$. The first singular value is one and its corresponding left singular function is $\varphi_1(x) = 1$. The second singular value of the valuation operator is $\sqrt{\lambda_1}$ and the corresponding left singular function is $\varphi_2(x) = z_1^P(x)$. The next three singular values are given by $\sqrt{\lambda_2}$, $\lambda_1$, and $\sqrt{\lambda_1\lambda_2}$, and corresponding left singular functions are $\varphi_3(x) = z_2^P(x)$, $\varphi_4(x) = \frac{1}{\sqrt{2}}\left(\left(z_1^P(x)\right)^2 - 1\right)$, and

[13] Here, $b$ is determined by: $b = S_0 \times {}^{(1+m_g)^T}/_{a^*_{x+T}} = S_0 \times {}^{(1+m_g)^T({}_TE_x)}/_{(\sum_{k=1}^{\infty}{}_{T+k}E_x)}$, where $m_g$ is the guaranteed rate of return and $a^*_{x+T}$ is actuarial present value based on forward rates. We set $m_g = 2\%$. With above parameter values, the probability that $S_T > b\,a_{x+T}(T)$ is approximately 40%.

$\varphi_5(x) = z_1^P(x)z_2^P(x)$. In contrast, a naïve choice of five monomials may result in the sequence $(1, q_\tau, r_\tau, \mu_{x+\tau}, q_\tau^2)$ or another arbitrary arrangement.

We implement the LSM approximation to the capital variable and vary the number of basis functions. In Figure 5, we provide empirical densities based on $N = 3,000,000$ and approximate realizations calculated via the LSM algorithm for different numbers of basis functions $M$. Here we rely on the optimal basis functions from Proposition 4.3 (Hermite polynomials). As is evident from the figure, the required number of basis function is relatively large compared to the univariate case from the previous section.[14] The approximation becomes closer to the exact density as $M$ increases.

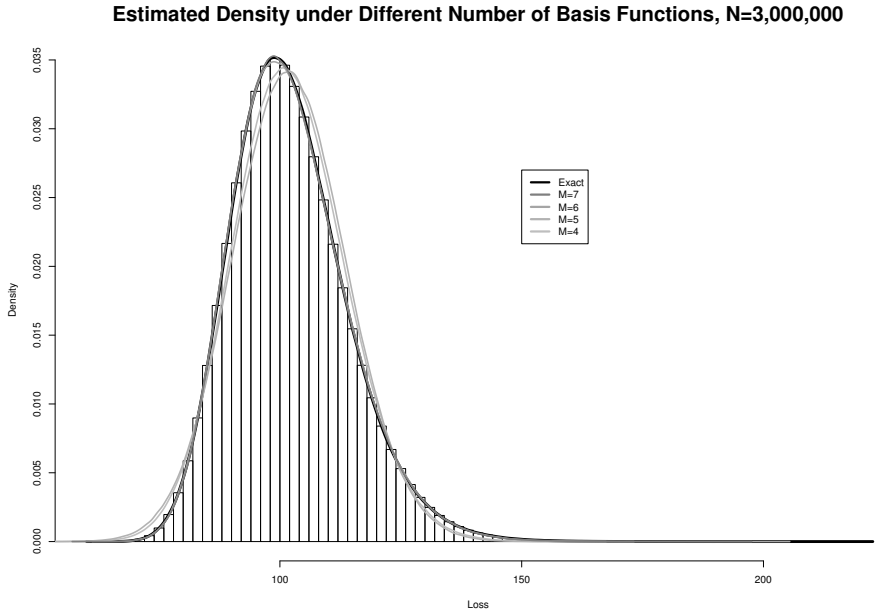



Figure 5: Empirical densities of $V(\tau)$ based on $N = 3,000,000$ Monte Carlo realizations; exact and using the LSM algorithm with $M$ singular functions in the approximation.

To assess the performance of optimal basis functions relative to naïve choices, in Table 2 we report statistical differences to the exact distribution according to various statistical distance measures for singular functions (left column) and simple monomials (right column).[15] We find that the optimal basis functions perform uniformly better than the simple polynomials. Furthermore, the table demonstrates that in this higher-dimensional setting, the functional approximation is more relevant than in the univariate setting in the previous section. More precisely, here we observe improvements in the statistical measures when using more basis functions even when keeping the number of simulations constant.

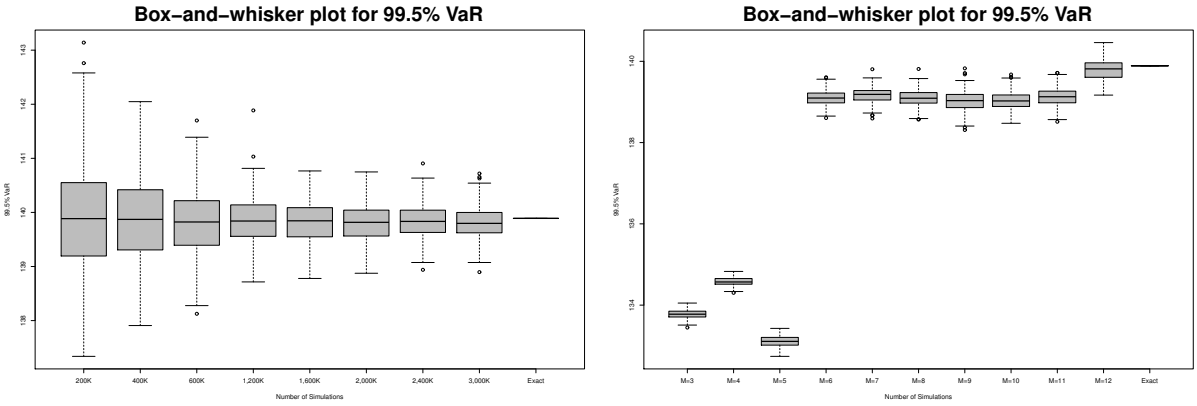
Moving to the calculation of the company's capital requirement, Figure 6 plots estimates for VaR at 99.5% (a) using a fixed number of (optimal) basis functions and varying the number of simulations, and (b) using a fixed number of simulations and varying the number of basis functions (box plots based on 300 runs). Similarly to the previous section, Figure 6a displays that the dispersion of the distribution of VaR is decreasing as $N$ increases. However, for $N = 3,000,000$, the bulk of the estimates are located between 139.11 and 140.61, which covers 95% of VaR estimates

[14] Since it is impossible to obtain the exact loss distribution at the risk horizon, we consider the estimated loss distribution obtained from the LSM algorithm with $M = 34$ monomials and $N = 30 \times 10^6$ simulations as "exact."

[15] Here, the set of monomial basis functions when $M = 6$ in Table 2 is $(1, q_\tau, r_\tau, \mu_{x+\tau}, q_\tau^2, r_\tau^2)$. For $M = 12$ we include all second-order terms.

| Order | | Singular Functions | Simple Monomials |
|---|---|---|---|
| | KS | $2.426 \times 10^{-2}$ | $2.776 \times 10^{-2}$ |
| $M = 4$ | KL | $2.151 \times 10^{-4}$ | $2.304 \times 10^{-4}$ |
| | JS | $7.400 \times 10^{-3}$ | $7.659 \times 10^{-3}$ |
| | KS | $1.990 \times 10^{-3}$ | $5.560 \times 10^{-3}$ |
| $M = 6$ | KL | $4.591 \times 10^{-6}$ | $4.232 \times 10^{-5}$ |
| | JS | $1.071 \times 10^{-3}$ | $3.261 \times 10^{-3}$ |
| | KS | $1.481 \times 10^{-3}$ | $1.708 \times 10^{-3}$ |
| $M = 12$ | KL | $7.137 \times 10^{-7}$ | $1.285 \times 10^{-6}$ |
| | JS | $4.165 \times 10^{-4}$ | $5.644 \times 10^{-4}$ |

Table 2: Statistical distances between the empirical density function based on the exact realizations and the LSM approximation using different basis functions; mean of three-hundred simulations with $N = 3,000,000$.




(a) $M = 12$ (b) $N = 3,000,000$.

Figure 6: Box-and-whisker diagrams for 99.5% VaR calculated using the LSM algorithm with different number of simulations $N$ and a fixed number of basis functions (a); and with different number of basis functions $M$ and a fixed number of simulations (b).

and safely contains the correct estimate – illustrating the viability of the approach. We observe a slight downward trend in the mean of the $\text{VaR}_{99.5\%}$ in line with the positive bias result from Proposition 3.4. In Figure 6b, we plot the distribution of 99.5% VaR for $N = 3,000,000$ and different choices for number of basis functions $M$. We see that a small number of basis functions, e.g. $M = 3$ or $M = 5$, can lead to a severe misestimation. As we increase the number of basis functions, the estimated 99.5% VaRs converges to the exact 99.5% VaR, although the distribution becomes more dispersed. Again, this emphasizes the relevance of the joint behavior as $N$ and $M$ increase.

To analyze the viability of naïve choices, in Table 3, we compare the performance of six optimal basis functions to various combinations of six simple polynomial basis functions. In particular, we choose a constant term and first-order terms in each variables, and we then consider eight choices for the remaining two terms. Again, we observe that the singular functions provide a uniformly better fit than the polynomials. Furthermore, we notice that a poor choice in the basis function (Combinations 5-8) lead to a severe underestimation of the VaR at 99.5%, where it appears that

| | KS | KL | JS | $\text{VaR}_{99.5\%}$ |
|---|---|---|---|---|
| Singular | $1.943 \times 10^{-3}$ | $4.601 \times 10^{-6}$ | $1.072 \times 10^{-3}$ | 139.09 |
| Comb. 1 ($q_\tau^2$, $r_\tau^2$) | $5.600 \times 10^{-3}$ | $4.233 \times 10^{-5}$ | $3.262 \times 10^{-3}$ | 140.11 |
| Comb. 2 ($q_\tau^2$, $\mu_{x+\tau}^2$) | $9.447 \times 10^{-3}$ | $2.638 \times 10^{-5}$ | $2.569 \times 10^{-3}$ | 140.57 |
| Comb. 3 ($q_\tau^2$, $q_\tau r_\tau$) | $3.910 \times 10^{-3}$ | $3.056 \times 10^{-5}$ | $2.772 \times 10^{-3}$ | 139.83 |
| Comb. 4 ($q_\tau^2$, $q_\tau \mu_{x+\tau}$) | $2.751 \times 10^{-3}$ | $3.218 \times 10^{-5}$ | $2.841 \times 10^{-3}$ | 139.12 |
| Comb. 5 ($r_\tau^2$, $q_\tau r_\tau$) | $2.444 \times 10^{-2}$ | $2.250 \times 10^{-4}$ | $7.566 \times 10^{-3}$ | 134.57 |
| Comb. 6 ($r_\tau^2$, $q_\tau \mu_{x+\tau}$) | $3.671 \times 10^{-2}$ | $2.115 \times 10^{-4}$ | $7.334 \times 10^{-3}$ | 132.35 |
| Comb. 7 ($\mu_{x+\tau}^2$, $q_\tau r_\tau$) | $2.117 \times 10^{-2}$ | $2.083 \times 10^{-4}$ | $7.282 \times 10^{-3}$ | 135.28 |
| Comb. 8 ($\mu_{x+\tau}^2$, $q_\tau \mu_{x+\tau}$) | $3.298 \times 10^{-2}$ | $1.969 \times 10^{-4}$ | $7.079 \times 10^{-3}$ | 133.08 |

Table 3: Statistical distances between the empirical density function based on the exact realizations and the LSM approximation using different combinations, and VaR at 99.5%; mean of three-hundred simulations with $N = 3,000,000$.

omitting higher-order terms in $q_\tau$ is the key issue.

# 6 Conclusion

We discuss a Least-Squares Monte Carlo (LSM) algorithm for estimating risk measures in "nested" settings. The algorithm relies on functional approximations of conditional expected values and least-squares regression. After establishing the algorithm, we analyze convergence of the approach and examine properties when estimating VaR. Moreover, we discuss the choice of basis functions in the functional approximation. Specifically, we show that, under certain conditions, the left singular functions of the valuation operator that maps cash flows to capital present *optimal* basis functions for a model framework. We derive optimal basis functions in settings where the underlying Markov state variable follows a Gaussian distribution, and we apply our ideas in two relevant examples from life insurance.

Our numerical illustrations document that the algorithm can provide viable results at relatively low computational costs. The algorithm, therefore, provides one potential solution to pressing practical problems such as the calculation of capital requirements in life insurance according to the recent Solvency II directive. Two key insights emerge from our analyses in view of applying the LSM algorithm in practical settings. First, increasing the number of basis functions comes at a significant cost since it is necessary to simultaneously increase the number of simulations $N$. This is required to establish convergence in theory, since the number of simulations typically has to increase much faster; and also in our illustrations, the variance of the estimates increased markedly when adding in additional basis terms. Second, in multivariate settings, a key issue is not only choosing the functional class of basis functions – which appears less crucial in our exercises – but rather the combinations of basis functions that are important for spanning the payoff space in view of valuation. Even in the three-dimensional setting considered here, this is of critical importance as naïve choices may yield significantly worse results. The choice of basis functions will become even more important as the complexity and the dimensionality of the problem increase, as it is the case in practical applications.

# Appendix

## A Proofs

***Proof of Lemma 2.1.*** 1. Let $A \in \mathcal{F}_t,\ 0 \le t \le \tau$. Then:

$$\begin{aligned}\tilde{\mathbb{P}}(A) &= \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbf{1}_A] = \mathbb{E}^{\mathbb{P}}\left[\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}\mathbf{1}_A\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}\mathbf{1}_A\middle|\mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{\mathbf{1}_A}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\middle|\mathcal{F}_\tau\right]\right] = \mathbb{P}(A).\end{aligned}$$

2. Let $X : \Omega \to \mathbb{R}$ be a random variable. Then:

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}[X\,|\mathcal{F}_\tau] &= \underbrace{\frac{1}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}}_{=1}\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}X\middle|\mathcal{F}_\tau\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{X\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}\middle|\mathcal{F}_\tau\right] \\ &= \frac{1}{\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\,|\mathcal{F}_\tau\right]}\mathbb{E}^{\mathbb{P}}\left[\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}X\middle|\mathcal{F}_\tau\right] = \mathbb{E}^{\mathbb{Q}}\left[X|\,\mathcal{F}_\tau\right].\end{aligned}$$

□

***Proof of Lemma 2.2.*** Linearity is obvious. For the proof of continuity, consider a sequence $h^{(n)} \to h \in \mathcal{H}$. Then:

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[L\,h^{(n)} - L\,h\right]^2 \\ =\ &\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{j=\tau}^{T}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_j^{(n)} - h_j\right)(Y_j)\,|Y_\tau\right]\right)^2\right] \\ =\ &\mathbb{E}^{\mathbb{P}}\left[\sum_{j,k}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_j^{(n)} - h_j\right)(Y_j)\,|Y_\tau\right]\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_k^{(n)} - h_k\right)(Y_k)\,|Y_\tau\right]\right] \\ \le\ &\sum_{j,k}\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_j^{(n)} - h_j\right)(Y_j)\,|Y_\tau\right]\right)^2\right]}\times\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_k^{(n)} - h_k\right)(Y_k)\,|Y_\tau\right]\right)^2\right]} \\ \le\ &\sum_{j,k}\sqrt{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_j^{(n)} - h_j\right)^2(Y_j)\right]}\times\sqrt{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(h_k^{(n)} - h_k\right)^2(Y_k)\right]} \to 0,\ n\to\infty,\end{aligned}$$

where we used the Cauchy-Schwarz inequality, the conditional Jensen inequality, and the tower property of conditional expectations. □

***Proof of Lemma 2.3.*** Consider the operator $L^{(j)}$ mapping from $L^2\left(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{P}}_{Y_j}\right)$ to $L^2\left(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau}\right)$. Since $L^{(j)}$ is the (conditional) expectation under the assumption that there exists a joint density,

it can be represented as:

$$L^{(j)}\,\mathbf{x} = \int_{\mathbb{R}^d} \mathbf{x}(y)\,\pi_{Y_j|Y_\tau}(y|x)\,dy = \int_{\mathbb{R}^d} \mathbf{x}(y)\frac{\pi_{Y_\tau,Y_j}(x,y)}{\pi_{Y_\tau}(x)}\,dy$$
$$= \int_{\mathbb{R}^d} \mathbf{x}(y)\frac{\pi_{Y_\tau,Y_j}(x,y)}{\pi_{Y_j}(y)\pi_{Y_\tau}(x)}\pi_{Y_j}(y)\,dy = \int_{\mathbb{R}^d} \mathbf{x}(y)\,k(x,y)\,\pi_{Y_j}(y)\,dy,$$

where $\mathbf{x}$ is an element of $L^2\left(\mathbb{R}^d,\mathcal{B},\tilde{\mathbb{P}}_{Y_j}\right)$, $\pi_{Y_j}(y)$ and $\pi_{Y_\tau}(x)$ are marginal density functions for $Y_j$ and $Y_\tau$ in $L^2\left(\mathbb{R}^d,\mathcal{B},\tilde{\mathbb{P}}_{Y_j}\right)$ and $L^2\left(\mathbb{R}^d,\mathcal{B},\mathbb{P}_{Y_\tau}\right)$, respectively, and $k(x,y) = \frac{\pi_{Y_\tau,Y_j}(x,y)}{\pi_{Y_j}(y)\pi_{Y_\tau}(x)}$. Thus, $L^{(j)}$ is an integral operator with kernel $k(x,y)$. Moreover:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |k(x,y)|^2\,\pi_{Y_j}(y)\pi_{Y_\tau}(x)\,dy\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \pi_{Y_j|Y_\tau}(y|x)\,\pi_{Y_\tau|Y_j}(x|y)\,dy\,dx < \infty.$$

Thus, $L^{(j)}$ is a Hilbert-Schmidt operator (e.g. Prop. VI.6.3 in Werner (2005)), and therefore compact. Finally, $L$ is the sum of $L^{(j)}$, $j = \tau, ..., T$, and therefore also compact. □

***Proof of Proposition 3.1.*** $\mathbb{P}_{Y_\tau}$ is a regular Borel measure as a finite Borel measure and hence $L^2\left(\mathbb{R}^d,\mathcal{B},\mathbb{P}_{Y_\tau}\right)$ is separable (see Proposition I.2.14 and p. 33 in Werner (2005)). Now if $\{e_k,\ k = 1,2,\ldots,M\}$ are independent, by Gram-Schmidt we can find an orthonormal system $S = \{f_k,\ k = 1,2,\ldots,M\}$ with $\operatorname{lin}\{e_k,\ k = 1,2,\ldots,M\} = \operatorname{lin} S$.[16] For $S$, on the other hand, we can find an orthonormal basis $\{f_k,\ k \in \mathbb{N}\} = S' \supset S$. Hence:

$$\widehat{C}_\tau^{(M)} = \sum_{k=1}^{M} \alpha_k\,e_k = \sum_{k=1}^{M} \underbrace{\tilde{\alpha}_k}_{\langle C_\tau, f_k\rangle}\ f_k \to \sum_{k=1}^{\infty} \tilde{\alpha}_k\,f_k = C_\tau,\ M \to \infty,$$

where:

$$\left\|\widehat{C}_\tau^{(M)} - C_\tau\right\|^2 = \sum_{k=M+1}^{\infty} |\langle C_\tau, f_k\rangle|^2 \to 0,\ M \to \infty,$$

by Parseval's identity.

For the second part, we note that:

$$(\hat{\alpha}_1^{(N)},\ldots,\hat{\alpha}_M^{(N)})' = \hat{\alpha}^{(N)} = \left(A^{(M,N)}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N} e\left(Y_\tau^{(i)}\right)\,V_\tau^{(i)},$$

where $e(\cdot) = (e_1(\cdot),\ldots,e_M(\cdot))'$ and $A^{(M,N)} = \left[\frac{1}{N}\sum_{i=1}^{N} e_k(Y_\tau^{(i)})\,e_l(Y_\tau^{(i)})\right]_{1\le k,l\le M}$ is invertible for large enough $N$ since we assume that the basis functions are linearly independent. Hence:

$$\hat{\alpha}^{(N)} \to \alpha = (\alpha_1,\ldots,\alpha_M)' = \left(A^{(M)}\right)^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}\left[e\left(Y_\tau\right)\left(\sum_{k=\tau}^{T} x_k\right)\right]\ \tilde{\mathbb{P}}\text{-a.s.},$$

by the law of large numbers, where $A^M = \left[\mathbb{E}^{\tilde{\mathbb{P}}}\left[e_k\left(Y_\tau\right)\,e_l\left(Y_\tau\right)\right]\right]_{1\le k,l\le M}$, so that:

$$\widehat{C}_\tau^{(M,N)} = e'\,\hat{\alpha}^{(N)} \to e'\alpha = \widehat{C}_\tau^{(M)}\ \tilde{\mathbb{P}}\text{-a.s.}$$

[16] We denote by $\operatorname{lin} S$ the (sub-)space spanned by the elements of $S$.

Finally, for the third part, write:

$$V_\tau^{(i)} = \sum_{k=\tau}^{T} x_k \left(Y_\tau^{(i)}\right) = \sum_{j=1}^{M} \alpha_j e_j \left(Y_\tau^{(i)}\right) + \epsilon_j,$$

where $\mathbb{E}\left[\epsilon_j | Y_\tau\right] = 0, \mathrm{Var}\left[\epsilon_j | Y_\tau\right] = \Sigma(Y_\tau)$, and $\mathrm{Cov}\left[\epsilon_i, \epsilon_j | Y_\tau\right] = 0$. Thus (see e.g. Section 6.13 in Amemiya (1985)):

$$\sqrt{N}[\alpha - \hat{\alpha}^{(N)}] \longrightarrow \text{Normal}\left[0, \underbrace{\left(A^{(M)}\right)^{-1}\left[\mathbb{E}^{\mathbb{P}}\left[e_k(Y_\tau)e_l(Y_\tau)\Sigma(Y_\tau)\right]\right]_{1\le k,l\le M}\left(A^{(M)}\right)^{-1}}_{\tilde{\xi}}\right],$$

so that:

$$\sqrt{N}\left[\widehat{C}_\tau^{(M)} - \widehat{C}_\tau^{(M,N)}\right] = e'[\alpha - \hat{\alpha}^{(N)}]\sqrt{N} \longrightarrow \text{Normal}\ (0, \xi^{(M)}),$$

where:

$$\xi^{(M)} = e'\,\tilde{\xi}\,e. \tag{30}$$

□

***Proof of Corollary 3.1.*** Relying on the notation from the proof of Proposition 3.1, we now have:

$$\hat{\alpha}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} e\left(Y_\tau^{(i)}\right)\ V_\tau^{(i)} \to \alpha,\ N \to \infty$$

in $L^2\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right)$ by the $L^2$-version of the weak law of large numbers (Durett, 1996). Thus:

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left|e(Y_\tau)'\,\hat{\alpha}^{(N)} - e(Y_\tau)'\,\alpha\right|\right] &\le \sum_{k=1}^{M}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left|e_k(Y_\tau)'\,\left(\hat{\alpha}_k^{(N)} - \alpha_k\right)\right|\right] \\ &\le \sum_{k=1}^{M}\sqrt{\mathbb{E}^{\tilde{\mathbb{P}}}\left[e_k^2(Y_\tau)\right]}\sqrt{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\hat{\alpha}_k^{(N)} - \alpha_k\right]^2} \to 0,\ N \to \infty.\end{aligned}$$

The last assertion in the statement is a direct consequence of the Extended Namioka Theorem in Biagini and Fritelli (2009). □

***Proof of Proposition 3.2.*** Since $(V_\tau^{(i)}, Y_\tau^{(i)})$ are i.i.d. as Monte Carlo trials, the first part of Assumption 1 in Newey (1997) is automatically satisfied. The conditions in the proposition are then exactly Assumptions 1 (part 2), 2, and 3 in his paper for $d = 0$. Thus, the claim follows by the first part of Theorem 1 in Newey (1997). □

***Proof of Proposition 3.3.*** Analogously to the proof of Proposition 3.2, the first part of Assumption 1 in Newey (1997) is automatically satisfied. The conditions in the proposition are taken from the second part of Assumption 1, Assumption 8, the discussion following Assumption 8, and Assumption 9 in his paper. Thus, the claim follows by the first part of Theorem 4 in Newey (1997). □

***Proof of Corollary 3.2.*** The first assertion immediately follows from convergence in distribution as discussed in Section 3.1. For the quantiles, the convergence for all continuity points of $F_{C_\tau}^{-1}$ follows from Proposition 3.1 and the standard proof of Skorokhod's representation theorem (see e.g. Lemma 1.7 in Whitt (2002)). □

***Regularity Conditions in Proposition 3.4.*** *(Gordy and Juneja, 2010).* Regularity conditions on the joint probability function (pdf) $g$ of $(-\widehat{C}_\tau^{(M)}, Z^{(N)})$:

- The joint pdf $g_N(\cdot,\cdot)$, its partial derivatives $\frac{\partial}{\partial y} g_N(y,z)$, and $\frac{\partial^2}{\partial y^2} g_N(y,z)$ exist for each $N$ and for all $(y,z)$.
- For $N \geq 1$, there exist non-negative functions $p_{0,N}(\cdot)$, $p_{1,N}(\cdot)$, and $p_{2,N}(\cdot)$ such that:
  - $g_N(y,z) \leq p_{0,N}(z)$,
  - $\left|\frac{\partial}{\partial y} g_N(y,z)\right| \leq p_{1,N}(z)$,
  - $\left|\frac{\partial^2}{\partial y^2} g_N(y,z)\right| \leq p_{2,N}(z)$, and

  for all $y$ and $z$. In addition:
  $$\sup_N \int_{-\infty}^{\infty} |z|^r p_{i,N}(z) dz < \infty$$
  for $i = 0, 1, 2$ and $0 \leq r \leq 4$.

The proof of Proposition 3.4 directly follows Proposition 2 in Gordy and Juneja (2010). □

***Singular Value Decomposition of a Compact Operator (Section 4.2).*** Suppose the operator $A$ mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$ is compact, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces. Then, $A$ can be represented in the following form (see Section VI.3 in Werner (2005) or Huang (2012)):

$$A\mathbf{x} = \sum_{k=1}^{\infty} \lambda_k \langle \mathbf{x}, g_k \rangle_{\mathcal{H}_1} f_k, \tag{31}$$

where:

- $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ denotes the inner product in $\mathcal{H}_1$;
- $\{\lambda_k^2\}$ are non-zero eigenvalues of $A^*A$ and $AA^*$ with $\lambda_1 \geq \lambda_2 \geq \cdots$, counted according to their multiplicity. Here, $\lambda_k$ is called the $k$-th singular value of $A$;
- $\{g_k\} \subset \mathcal{H}_1$, called the (right) *singular functions*, are the orthonormal eigenfunctions of $A^*A$; and
- $\{f_k\} \subset \mathcal{H}_2$, called the (left) *singular functions*, are the orthonormal eigenfunctions of $AA^*$ satisfying $A\, g_k = \lambda_k\, f_k$.

The representation (31) is called *singular value decomposition* (SVD) of $A$ and the triple $(\lambda_k, g_k, f_k)$ is called *singular system* for $A$. The functional sequences, $\{g_k\}_{k\geq 1}$ and $\{f_k\}_{k\geq 1}$, form complete orthonormal sequences of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The singular values $\lambda_k$ are non-negative and the only possible accumulation point is zero. □

***Proof of Proposition 4.1.*** We consider the approximation of $L$ by an arbitrary rank-$M$ operator $L_F$, which can be represented as:

$$L_F = \sum_{k=1}^{M} \alpha_k \langle \cdot\,, u_k \rangle\, e_k,$$

where $\{\alpha_k\}_{k=1}^M \subseteq \mathbb{R}_+$, $\{u_k\}_{k=1}^M$ are orthonormal in $\mathcal{H}$, and $\{e_k\}_{k=1}^M$ are orthonormal in $L^2\left(\mathbb{R}^d, \mathcal{B}, \mathbb{P}_{Y_\tau}\right)$. Denote by $L_F^*$ the operator when choosing $(\alpha_k, u_k, e_k) = (\omega_k, s_k, \varphi_k)$. Then:

$$\begin{aligned}
\inf_{L_F} \|L - L_F\|^2 &\leq \sup_{\|\mathbf{x}\|=1} \|L\mathbf{x} - L_F^*\mathbf{x}\|^2 \\
&= \sup_{\|\mathbf{x}\|=1} \left\| \sum_{k=M+1}^{\infty} \omega_k \langle \mathbf{x}, s_k \rangle \varphi_j \right\|^2 \\
&= \sup_{\|\mathbf{x}\|=1} \sum_{k=M+1}^{\infty} \omega_k^2 \langle \mathbf{x}, s_k \rangle^2 \;=\; \omega_{M+1}^2.
\end{aligned}$$

On the other hand, consider any alternative system $(\alpha_k, u_k, e_k)$ for an arbitrary finite-rank operator $L_F$. Then choose a non-zero $\mathbf{x}_0$ such that $\mathbf{x}_0 \in \operatorname{lin}\{s_1, ..., s_{M+1}\} \cap \operatorname{lin}\{u_1, ..., u_M\}^\perp \neq \{0\}$. Note that $L - L_F$ is compact and bounded. Therefore:

$$\begin{aligned}
\|L - L_F\|^2 \geq \frac{\|L\mathbf{x}_0 - L_F\,\mathbf{x}_0\|^2}{\|\mathbf{x}_0\|^2} &= \frac{\|L\mathbf{x}_0\|^2}{\|\mathbf{x}_0\|^2} \\
= \frac{\sum_{k=1}^{M+1} \omega_k^2 |\langle \mathbf{x}_0, s_k \rangle|^2}{\sum_{k=1}^{M+1} |\langle \mathbf{x}_0, s_k \rangle|^2} &\geq \omega_{M+1}^2.
\end{aligned}$$

Hence:

$$\inf_{L_F} \|L - L_F\|^2 = \omega_{M+1}^2 = \|L - L_F^*\|.$$

Now since:

$$\inf_{L_F} \|L - L_F\|^2 = \inf_{\{e_1, ..., e_M\}} \|L - P(e_1, ..., e_M) \cdot\, L\|^2,$$

where $P(e_1, ..., e_M)$ denotes the orthogonal projection on the subspace spanned by $(e_1, ..., e_M)$, the claim follows by Equation (7). □

***Proof of Proposition 4.2.*** Proceeding as in Equation (9) and with Equation (7), we obtain:

$$\begin{aligned}
\inf_{\alpha_M} \sup_{y \in \mathcal{Y}} \left| C_\tau(y) - \sum_{k=1}^{M} \alpha_{M,k}\, e_k(y) \right| &\leq \sup_{y \in \mathcal{Y}} \left| C_\tau(y) - \widehat{C}_\tau^{(M)}(y) \right| \\
&= \sup_{y \in \mathcal{Y}} | \sum_{k=M+1}^{\infty} \omega_k \langle \mathbf{x}, s_k \rangle\, \varphi_k(y)| \\
&\leq \sum_{k=M+1}^{\infty} \omega_k\, |\langle \mathbf{x}, s_k \rangle| \sup_{y \in \mathcal{Y}} |\varphi_k(y)| \\
&\leq \sum_{k=M+1}^{\infty} \omega_k\, \|\mathbf{x}\|\, \|s_k\| \sup_{y \in \mathcal{Y}} |\varphi_k(y)| \\
&= \sum_{k=M+1}^{\infty} \omega_k\, \|\mathbf{x}\| \sup_{y \in \mathcal{Y}} |\varphi_k(y)| = O\left(\omega_M\right)
\end{aligned}$$

for a fixed $\mathbf{x}$ since the $\{\varphi_k\}$ are uniformly bounded, where the second and third inequalities follow by the triangle and Cauchy-Schwarz inequalities, respectively.

Then, going through the assumptions of Proposition 3.2 with $B = I$ and $e^{(M)} = (e_1, ..., e_M)'$, we obtain:

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\tilde{e}^{(M)}(Y_\tau)\tilde{e}^{(M)}(Y_\tau)'\right] = I$$

due to the orthonormality of the singular functions. Therefore, the smallest eigenvalues is bounded away from zero uniformly for every $M$. Moreover, for fixed $y \in \mathcal{Y}$, $||\tilde{e}^{(M)}(y)|| = \sqrt{\varphi_1(y)^2 + \cdots \varphi_M(y)^2}$, so that:

$$\begin{aligned}\sup_{y\in\mathcal{Y}} ||\tilde{e}^{(M)}(y)|| =& \sup_{y\in\mathcal{Y}} \sqrt{\varphi_1(y)^2 + \cdots \varphi_1(y)^2} \\ \leq& \sqrt{\sum_{k=1}^{M} \sup_{y\in\mathcal{Y}} \varphi_k(y)^2} \leq \sqrt{\max_{1\leq k\leq M} \sup_{y\in\mathcal{Y}} \varphi_k(y) \cdot M} = C\sqrt{M} = \zeta_0(M)\end{aligned}$$

since the $\{\varphi_k\}$ are uniformly bounded. Thus, the claim follows by Proposition 3.2. □

***Proof of Lemma 4.1.*** The assertions on the conditional distributions are standard. For showing that $L$ is compact, we check that the transition and the reverse transition density functions satisfy the condition in Lemma 2.3. Note that the transition density function can be written as:

$$\begin{aligned}\pi_{Y_T|Y_\tau}(y|x) &= g(y; \mu_T + \Gamma'\Sigma_\tau^{-1}(x - \mu_\tau), \Sigma_{T|\tau}) \\ &= \frac{1}{(2\pi)^{d/2}|\Sigma_{T|\tau}|^{1/2}} \exp\left[-\frac{1}{2}\left(y - \mu_T - \Gamma'\Sigma_\tau^{-1}(x - \mu_\tau)\right)' \Sigma_{T|\tau}^{-1}\left(y - \mu_T - \Gamma'\Sigma_\tau^{-1}(x - \mu_\tau)\right)\right] \\ &= \frac{1}{(2\pi)^{d/2}|\Sigma_{T|\tau}|^{1/2}} \frac{\left|\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau\right|^{1/2}}{\left|\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau\right|^{1/2}} \\ &\quad \times \exp\left[-\frac{1}{2}\left(x - \mu_\tau - \Sigma_\tau(\Gamma')^{-1}(y - \mu_T)\right)' \Sigma_\tau^{-1}\Gamma\Sigma_{T|\tau}^{-1}\Gamma'\Sigma_\tau^{-1}\left(x - \mu_\tau - \Sigma_\tau(\Gamma')^{-1}(y - \mu_T)\right)\right] \\ &= \frac{|\Sigma_\tau|}{|\Gamma|} g\left(x; \mu_\tau + \Sigma_\tau(\Gamma')^{-1}(y - \mu_T), \Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau\right).\end{aligned}$$

We evaluate the following integral:

$$\begin{aligned}&\int_{\mathbb{R}^d} \pi_{Y_T|Y_\tau}(y|x)\pi_{Y_\tau|Y_T}(x|y)dx \\ &= \frac{|\Sigma_\tau|}{|\Gamma|}\int_{\mathbb{R}^d} g\left(x; \mu_\tau + \Sigma_\tau(\Gamma')^{-1}(y - \mu_T), \Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau\right) \\ &\qquad \times g\left(x; \mu_\tau + \Gamma\Sigma_\tau^{-1}(y - \mu_T), \Sigma_{\tau|T}\right) dx \\ &= \frac{|\Sigma_\tau|}{|\Gamma|(2\pi)^{d/2}} \frac{1}{\left|\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau + \Sigma_{\tau|T}\right|^{1/2}} \\ &\quad \times \exp\Bigg[-\frac{1}{2}\left(\Sigma_\tau(\Gamma')^{-1}(y - \mu_T) - \Gamma\Sigma_T^{-1}(y - \mu_T)\right)' \left(\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau + \Sigma_{\tau|T}\right)^{-1} \\ &\qquad \times \left(\Sigma_\tau(\Gamma')^{-1}(y - \mu_T) - \Gamma\Sigma_T^{-1}(y - \mu_T)\right)\Bigg] \\ &= \frac{|\Sigma_\tau|}{|\Gamma|(2\pi)^{d/2}} \frac{1}{\left|\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau + \Sigma_{\tau|T}\right|^{1/2}} \\ &\quad \times \exp\Bigg[-\frac{1}{2}(y - \mu_T)' \underbrace{\left(\Gamma^{-1}\Sigma_\tau - \Sigma_T^{-1}\Gamma'\right)\left(\Sigma_\tau(\Gamma')^{-1}\Sigma_{T|\tau}\Gamma^{-1}\Sigma_\tau + \Sigma_{\tau|T}\right)^{-1}\left(\Sigma_\tau(\Gamma')^{-1} - \Gamma\Sigma_T^{-1}\right)}_{V^{-1}}(y - \mu_T)\Bigg] \\ &= C_1 \times g(y; \mu_T, V),\end{aligned}$$

where we use results on the product of Gaussian densities (Vinga, 2004) and where $C_1$ is an appropriate constant to obtain $g(y;\mu_T,V)$. Therefore:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\pi_{Y_T|Y_\tau}(y|x)\pi_{Y_\tau|Y_T}(x|y)\,dx\,dy=\int_{\mathbb{R}^d}C_1 g(y;\mu_T,V)\,dy=C_1<\infty.$$

□

***Proof of Lemma 4.2.*** $L^*$ can be found via:

$$\begin{aligned}\langle Lh,m\rangle_{\pi_{Y_\tau}}&=\int_{\mathbb{R}^d}Lh(x)\,m(x)\,\pi_{Y_\tau}(x)\,dx=\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}h(y)\pi_{Y_T|Y_\tau}(y|x)\,dy\right]m(x)\,\pi_{Y_\tau}(x)\,dx\\&=\int_{\mathbb{R}^d}h(y)\left[\int_{\mathbb{R}^d}m(x)\pi_{Y_\tau|Y_T}(x|y)\,dx\right]\pi_{Y_T}(y)\,dy=\langle h,L^*m\rangle_{\pi_{Y_T}},\end{aligned}$$

where $L^*m(y)=\int_{\mathbb{R}^d}m(x)\pi_{Y_\tau|Y_T}(x|y)\,dx$. We obtain for $LL^*$:

$$\begin{aligned}LL^*\varphi(x)&=\int_{\mathbb{R}^d}L^*\varphi(s)\pi_{Y_T|Y_\tau}(s|x)\,ds\\&=\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\varphi(y)\pi_{Y_\tau|Y_T}(y|s)\,dy\right]\pi_{Y_T|Y_\tau}(s|x)\,ds\\&=\int_{\mathbb{R}^d}\varphi(y)\underbrace{\int_{\mathbb{R}^d}\pi_{Y_\tau|Y_T}(y|s)\pi_{Y_T|Y_\tau}(s|x)\,ds}_{K_A(x,y)}\,dy.\end{aligned}$$

It is useful to express the reverse density as in the proof of Lemma 4.1:

$$g(y;\mu_{Y_\tau|s},\Sigma_{\tau|T})=\frac{|\Sigma_T|}{|\Gamma|}g\left(s;\mu_T+\Sigma_T\Gamma^{-1}(y-\mu_\tau),\ \Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T\right).$$

Hence:

$$\begin{aligned}K_A(x,y)&=\int_{\mathbb{R}^d}\pi_{Y_\tau|Y_T}(y|s)\pi_{Y_T|Y_\tau}(s|x)\,ds\\&=\frac{|\Sigma_T|}{|\Gamma|}\int_{\mathbb{R}^d}g\left(s;\mu_T+\Sigma_T\Gamma^{-1}(y-\mu_\tau),\ \Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T\right)\times g(s;\mu_{T|x},\Sigma_{T|\tau})\,ds\\&=\frac{|\Sigma_T|}{|\Gamma|}\times\frac{1}{(2\pi)^{d/2}\left|\Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T+\Sigma_{T|\tau}\right|^{1/2}}\\&\quad\times\exp\Bigg(-\frac{1}{2}\left(\Sigma_T\Gamma^{-1}(y-\mu_\tau)-\Gamma'\Sigma_\tau^{-1}(x-\mu_\tau)\right)'\\&\qquad\times\left(\Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T+\Sigma_{T|\tau}\right)^{-1}\left(\Sigma_T\Gamma^{-1}(y-\mu_\tau)-\Gamma'\Sigma_\tau^{-1}(x-\mu_\tau)\right)\Bigg)\\&=\frac{1}{(2\pi)^{d/2}\left|\Gamma\Sigma_T^{-1}\left(\Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T+\Sigma_{T|\tau}\right)\Sigma_T^{-1}\Gamma'\right|^{1/2}}\\&\quad\times\exp\Bigg(-\frac{1}{2}\left(y-\mu_\tau-\Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}(x-\mu_\tau)\right)'(\Gamma^{-1})'\Sigma_T\left(\Sigma_T\Gamma^{-1}\Sigma_{\tau|T}(\Gamma')^{-1}\Sigma_T+\Sigma_{T|\tau}\right)^{-1}\\&\qquad\times\Sigma_T\Gamma^{-1}\left(y-\mu_\tau-\Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}(x-\mu_\tau)\right)\Bigg)\\&=g\big(y;\mu_\tau+\underbrace{\Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}}_{A}(x-\mu_\tau),\,\Sigma_\tau-\Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}\Gamma\Sigma_T^{-1}\Gamma'\big)\\&=g\big(y;\underbrace{\mu_\tau+A(x-\mu_\tau)}_{\mu_A(x)},\,\underbrace{\Sigma_\tau-A\Sigma_\tau A'}_{\Sigma_A}\big)=g\big(y;\mu_A(x),\Sigma_A\big),\end{aligned}$$

where in the third equality we again rely on results on the products of Gaussian densities from Vinga (2004). $L^*L$ can be derived analogously. □

***Proof of Lemma 4.3.*** We start by recalling the considerations from Khare and Zhou (2009): Let $(X_t)$ on $\mathbb{R}^d$ be a MAR(1) process satisfying the following stochastic difference equation:

$$X_t = \Phi X_{t-1} + \eta_t, \quad t \geq 1, \tag{32}$$

where $\Phi \in \mathbb{R}^{d\times d}$ and $(\eta_t)_{t\geq 1}$ are independent and identically distributed, $\eta_1 \sim N(0, H)$. $(X_t)$ has a unique stationary distribution $N(0, \Sigma)$ if and only if $H = \Sigma - \Phi\Sigma\Phi'$, and the process is reversible if and only if $\Phi\Sigma = \Sigma\Phi'$. Khare and Zhou (2009) show that if these assumptions are satisfied, the transformed Markov operator for (32) has eigenvalues which are products of eigenvalues of $\Phi$ and the corresponding eigenfunctions are products of Hermite polynomials.

Now note that for a random variable $Y$ that is distributed according to $K_A(x, \cdot)$, we can write:

$$Y - \mu_\tau = A(x - \mu_\tau) + \zeta_A, \tag{33}$$

where $\zeta \sim N(0, \Sigma_A)$. Since from Lemma 4.2 we have that $\Sigma_A = \Sigma_\tau - A\Sigma_\tau A'$ and:

$$A\Sigma_\tau = \Gamma\Sigma_T^{-1}\Gamma' = \Sigma_\tau\, A',$$

for $\Sigma = \Sigma_\tau$ the operator $L\,L^*$ has the same structure of the Markov operator for (32) that is reversible and stationary.

Following the approach by Khare and Zhou (2009), denote by $\Sigma_\tau^{1/2}$ the square root matrix of $\Sigma_\tau$. Then:

$$\Sigma_\tau^{-1/2} A \Sigma_\tau^{1/2} = \Sigma_\tau^{-1/2}\,\Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1/2}$$

is symmetric and thus orthogonally diagonalizable:

$$\Sigma_\tau^{-1/2} A \Sigma_\tau^{1/2} = P\Lambda P' \;\Leftrightarrow\; A = (\Sigma_\tau^{1/2}\,P)\,\Lambda\,(P'\Sigma_\tau^{-1/2}).$$

In particular, the entries of the diagonal matrix $\Lambda$ are the eigenvalues of $A$.

Now for the transformation (11) of the random vector $Y$ from (33), $z^P(Y)$, we obtain:

$$\begin{aligned}\mathbb{E}_{K_A}\left[z^P(Y)|x\right] &= P'\Sigma_\tau^{-1/2}A(x-\mu_\tau)\\ &= P'\Sigma_\tau^{-1/2}\Sigma_\tau^{1/2}P\Lambda P'\Sigma_\tau^{-1/2}(x-\mu_\tau) = \Lambda z^P(x),\end{aligned}$$

and:

$$\begin{aligned}\mathrm{Var}_{K_A}\left[z^P(Y)|x\right] &= P'\Sigma_\tau^{-1/2}\Sigma_A\Sigma_\tau^{-1/2}P\\ &= P'\Sigma_\tau^{-1/2}(\Sigma_\tau - A\Sigma_\tau A')\Sigma_\tau^{-1/2}P = I - \Lambda^2.\end{aligned}$$

Moreover:

$$\mathbb{E}_{\pi_{Y_\tau}}\left[z^P(Y_\tau)\right] = P'\Sigma_\tau^{-1/2}\mathbb{E}_{\pi_{Y_\tau}}\left[Y_\tau - \mu_\tau\right] = 0$$

and:

$$\mathrm{Var}_{\pi_{Y_\tau}}\left[z^P(Y_\tau)\right] = P'\Sigma_\tau^{-1/2}\Sigma_\tau\Sigma_\tau^{-1/2}P = I.$$

The second part follows analogously. □

***Proof of Proposition 4.3.*** For fixed $z_i^P(Y)$, we obtain from Carrasco and Florens (2011) that the univariate orthonormal Hermite polynomial of order $n_i$ is an eigenfunction under $K_A$ :

$$\mathbb{E}_{K_A}\left[h_{n_i}(z_i^P(Y))|x\right] = \lambda_i^{n_i} h_{n_i}(z_i^P(x)).$$

Moreover, the product of these polynomials are also eigenfunction since:

$$\mathbb{E}_{K_A}\left[\Pi_{i=1}^d h_{n_i}(z_i^P(Y))|x\right] = \Pi_{i=1}^d \mathbb{E}_{K_A}\left[h_{n_i}(z_i^P(Y))|x\right] = \left(\Pi_{i=1}^d \lambda_i^{n_i}\right)\left(\Pi_{i=1}^d h_{n_i}(z_i^P(x))\right).$$

The orthogonality of the eigenfunctions is proved in Khare and Zhou (2009). Note that the product of normalized Hermite polynomials is already normalized since:

$$\mathbb{E}_{\pi_{Y_\tau}}\left[\left(\Pi_{i=1}^d h_{n_i}(z_i^P(Y))\right)^2\right] = \mathbb{E}_{\pi_{Y_\tau}}\left[\Pi_{i=1}^d h_{n_i}\left(z_i^P\right)^2(Y)\right] = \Pi_{i=1}^d \mathbb{E}_{\pi_{Y_\tau}}\left[h_{n_i}\left(z_i^P(Y)\right)^2\right] = 1.$$

Right singular functions are obtained similarly from $z_i^Q(X)$. □

***Proof of Lemma 5.1.*** Under $\mathbb{P}$, we have:

$$r_\tau = r_0 e^{-\alpha\tau} + \gamma(1 - e^{-\alpha\tau}) + \sigma\int_0^\tau e^{-\alpha(\tau-t)}dW_t,$$

so that $r_\tau \sim N(\mu_\tau, \sigma_\tau^2)$ with $\mu_\tau = \gamma - (\gamma - r_0)e^{-\alpha\tau}$ and $\sigma_\tau^2 = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\tau})$.

Under $\mathbb{Q}_T$, we have:

$$r_T = r_\tau e^{-\alpha(T-\tau)} + M(\tau, T) + \sigma\int_\tau^T e^{-\alpha(T-t)}dZ_t^T,$$

where:

$$M(\tau, T) = \left(\bar{\gamma} - \frac{\sigma^2}{\alpha^2}\right)\left(1 - e^{-\alpha(T-\tau)}\right) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-\tau)}\right),$$

so that $r_T|r_\tau \sim N(\mu_{r_T|r_\tau}, \sigma^2_{r_T|r_\tau})$ with $\mu_{r_T|r_\tau} = r_\tau e^{-\alpha(T-\tau)} + M(\tau, T)$ and $\sigma^2_{r_{r_T|r_\tau}} = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-\tau)})$. Note that this distribution specifies the transition density of $r_T$ given $r_\tau$. The unconditional mean and variance of $r_T$ is given by:

$$\begin{aligned}\mu_T = \mathbb{E}^{\tilde{\mathbb{P}}}[r_T] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}_T}[r_T|r_\tau]\right] = \mathbb{E}^{\mathbb{P}}\left[r_\tau e^{-\alpha(T-\tau)} + M(\tau, T)\right] \\ &= \mu_\tau e^{-\alpha(T-\tau)} + M(\tau, T)\end{aligned}$$

and

$$\begin{aligned}\sigma_T^2 = \mathrm{Var}^{\tilde{\mathbb{P}}}[r_T] &= \mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{Q}_T}[r_T|r_\tau]\right] + \mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}_T}[r_T|r_\tau]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-\tau)}\right)\right] + \mathrm{Var}^{\mathbb{P}}\left[r_\tau e^{-\alpha(T-\tau)} + M(\tau, T)\right] \\ &= \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-\tau)}\right) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha\tau}\right)e^{-2\alpha(T-\tau)} = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha T}\right),\end{aligned}$$

so that $r_T \sim N(\mu_T, \sigma_T^2)$.

Moreover:

$$\begin{aligned}\mathrm{Cov}(r_\tau, r_T) &= \mathbb{E}^{\tilde{\mathbb{P}}}(r_T \cdot r_\tau) - \mu_\tau\mu_T = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}_T}[r_T \cdot r_\tau|r_\tau]\right] - \mu_\tau\mu_T \\ &= e^{-\alpha(T-\tau)}\sigma_\tau^2.\end{aligned}$$

Thus, we have for the joint distribution of $r_\tau$ and $r_T$:

$$\begin{bmatrix} r_\tau \\ r_T \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_\tau \\ \mu_T \end{bmatrix}, \begin{bmatrix} \sigma_\tau^2, & e^{-\alpha(T-\tau)}\sigma_\tau^2 \\ e^{-\alpha(T-\tau)}\sigma_\tau^2, & \sigma_T^2 \end{bmatrix} \right),$$

with $\rho = \rho_{r_\tau, r_T} = \text{Corr}(r_\tau, r_T) = e^{-\alpha(T-\tau)} \frac{1-e^{-2\alpha\tau}}{1-e^{-2\alpha T}}$. □

***Proof of Lemma 5.2.*** Under $\mathbb{P}$, the solutions of (22), (23), and (24) at time $\tau$ are:

$$\begin{aligned} q_\tau &= q_0 + \left(m - \frac{1}{2}\sigma_S^2\right)\tau + \sigma_S \int_0^\tau dW_s^S, \\ r_\tau &= r_0 e^{-\alpha\tau} + \gamma\left(1 - e^{-\alpha\tau}\right) + \sigma_r \int_0^\tau e^{-\alpha(\tau-t)} dW_t^r, \\ \mu_{x+\tau} &= \mu_x e^{\kappa\tau} + \psi \int_0^\tau e^{\kappa(\tau-u)} dW_u^\mu. \end{aligned}$$

Thus, the joint Gaussian distribution of $Y_\tau$ is given by:

$$\begin{bmatrix} q_\tau \\ r_\tau \\ \mu_{x+\tau} \end{bmatrix} \sim N\left( \begin{bmatrix} q_0 + \left(m - \frac{1}{2}\sigma_S^2\right)\tau \\ r_0 e^{-\alpha\tau} + \gamma\left(1 - e^{-\alpha\tau}\right) \\ \mu_x e^{\kappa\tau} \end{bmatrix}, \begin{bmatrix} \sigma_S^2\,\tau & \rho_{12}\sigma_S\sigma_r B_r(0,\tau) & \rho_{13}\sigma_S\psi B_\mu(0,\tau) \\ \rho_{12}\sigma_S\sigma_r B_r(0,\tau) & \sigma_r^2 \frac{1-e^{-2\alpha\tau}}{2\alpha} & \rho_{23}\sigma_r\psi\frac{1-e^{-(\alpha-\kappa)\tau}}{\alpha-\kappa} \\ \rho_{13}\sigma_S\psi B_\mu(0,\tau) & \rho_{23}\sigma_r\psi\frac{1-e^{-(\alpha-\kappa)\tau}}{\alpha-\kappa} & \psi^2\frac{e^{2\kappa\tau}-1}{2\kappa} \end{bmatrix} \right), \tag{34}$$

so that $\mu_\tau$ and $\Sigma_\tau$ are given by

$$\mu_\tau = \begin{bmatrix} q_0 + \left(m - \frac{1}{2}\sigma_S^2\right)\tau \\ r_0 e^{-\alpha\tau} + \gamma\left(1 - e^{-\alpha\tau}\right) \\ \mu_x e^{\kappa\tau} \end{bmatrix}, \quad \Sigma_\tau = \begin{bmatrix} \sigma_S^2\,\tau & \rho_{12}\sigma_S\sigma_r B_r(0,\tau) & \rho_{13}\sigma_S\psi B_\mu(0,\tau) \\ \rho_{12}\sigma_S\sigma_r B_r(0,\tau) & \sigma_r^2 \frac{1-e^{-2\alpha\tau}}{2\alpha} & \rho_{23}\sigma_r\psi\frac{1-e^{-(\alpha-\kappa)\tau}}{\alpha-\kappa} \\ \rho_{13}\sigma_S\psi B_\mu(0,\tau) & \rho_{23}\sigma_r\psi\frac{1-e^{-(\alpha-\kappa)\tau}}{\alpha-\kappa} & \psi^2\frac{e^{2\kappa\tau}-1}{2\kappa} \end{bmatrix}.$$

To derive the distribution under $\mathbb{Q}_E$, first note that for $\tau \le s < T$:

$$\begin{aligned} r_s =& e^{-\alpha(s-\tau)} r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(1 - e^{-\alpha(s-\tau)}\right) + \frac{\sigma_r^2}{2\alpha^2}\left(e^{-\alpha(T-s)} - e^{-\alpha(T+s-2\tau)}\right) \\ & - \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{e^{\kappa(T-s)} - e^{-\alpha(s-\tau)+\kappa(T-\tau)}}{\alpha-\kappa} - \frac{1 - e^{-\alpha(s-\tau)}}{\alpha}\right) + \sigma_r \int_\tau^s e^{-\alpha(s-y)} dZ_y^r, \end{aligned}$$

so that the integral of $\int_\tau^T r_s\, ds$ can be evaluated using the stochastic Fubini theorem:

$$\begin{aligned} \int_\tau^T r_s ds =& \left(\frac{1 - e^{-\alpha(T-\tau)}}{\alpha}\right) r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(T - \tau - \frac{1 - e^{-\alpha(T-\tau)}}{\alpha}\right) \\ & + \frac{\sigma_r^2}{2\alpha^2}\left(\frac{1 - e^{-\alpha(T-\tau)}}{\alpha} - \frac{e^{-\alpha(T-\tau)} - e^{-2\alpha(T-\tau)}}{\alpha}\right) \\ & - \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{e^{\kappa(T-\tau)} - 1}{\kappa(\alpha-\kappa)} - \frac{e^{\kappa(T-\tau)} - e^{-(\alpha-\kappa)(T-\tau)}}{\alpha(\alpha-\kappa)} - \frac{1}{\alpha}\left(T - \tau - \frac{1 - e^{-\alpha(T-\tau)}}{\alpha}\right)\right) \\ & + \sigma_r \int_\tau^T \frac{1 - e^{-\alpha(T-y)}}{\alpha} dZ_y^r. \end{aligned}$$

Thus, under $\mathbb{Q}_E$ with known $Y_\tau$, the solutions of (27), (28), and (29) are:

$$\begin{aligned}
q_T =& q_\tau + \left(\frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) \\
&+ \frac{\sigma_r^2}{2\alpha^2}\left(\frac{1-e^{-\alpha(T-\tau)}}{\alpha} - \frac{e^{-\alpha(T-\tau)} - e^{-2\alpha(T-\tau)}}{\alpha}\right) \\
&- \frac{1}{2}\sigma_S^2(T-\tau) - \frac{\rho_{12}\sigma_S\sigma_r}{\alpha}\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) - \frac{\rho_{13}\sigma_S\psi}{\kappa}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa} - T + \tau\right) \\
&- \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa(\alpha-\kappa)} - \frac{e^{\kappa(T-\tau)} - e^{-(\alpha-\kappa)(T-\tau)}}{\alpha(\alpha-\kappa)} - \frac{1}{\alpha}\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right)\right) \\
&+ \sigma_S\int_\tau^T dZ_s^S + \sigma_r\int_\tau^T \frac{1-e^{-\alpha(T-y)}}{\alpha} dZ_y^r, \\
r_T =& e^{-\alpha(T-\tau)} r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(1-e^{-\alpha(T-\tau)}\right) + \frac{\sigma_r^2}{2\alpha^2}\left(1-e^{-2\alpha(T-\tau)}\right) \\
&- \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa} - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) + \sigma_r\int_\tau^T e^{-\alpha(T-y)} dZ_y^r, \\
\mu_{x+T} =& e^{\kappa(T-\tau)}\mu_{x+\tau} - \frac{\psi^2}{\kappa}\left(\frac{e^{2\kappa(T-\tau)}-1}{2\kappa} - \frac{e^{\kappa(T-\tau)}-1}{\kappa}\right) - \frac{\rho_{23}\sigma_r\psi}{\alpha}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa} - \frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa}\right) \\
&+ \psi\int_\tau^T e^{\kappa(T-t)} dZ_t^\mu,
\end{aligned}$$

so that the (Gaussian) conditional distribution of $Y_T|Y_\tau$ is given by:

$$\begin{pmatrix} q_T \\ r_T \\ \mu_{x+T} \end{pmatrix} | Y_\tau \sim N\left(\begin{bmatrix} \mu_{q_T|q_\tau} \\ \mu_{r_T|r_\tau} \\ \mu_{\mu_{x+T}|\mu_{x+\tau}} \end{bmatrix}, \underbrace{\begin{bmatrix} \sigma^2_{q_T|q_\tau} & \sigma_{q_T,r_T|q_\tau,r_\tau} & \sigma_{q_T,\mu_{x+T}|q_\tau,\mu_{x+\tau}} \\ \sigma_{q_T,r_T|q_\tau,r_\tau} & \sigma^2_{r_T|r_\tau} & \sigma_{r_T,\mu_{x+T}|r_\tau,\mu_{x+\tau}} \\ \sigma_{q_T,\mu_{x+T}|q_\tau,\mu_{x+\tau}} & \sigma_{r_T,\mu_{x+T}|r_\tau,\mu_{x+\tau}} & \sigma^2_{\mu_{x+T}|\mu_{x+\tau}} \end{bmatrix}}_{\Sigma_{T|\tau}}\right), \tag{35}$$

where:

$$\begin{aligned}
\mu_{q_T|q_\tau} =& q_\tau + B_r(\tau,T) r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) \\
&+ \frac{\sigma_r^2}{2\alpha^2}\left(\frac{1-e^{-\alpha(T-\tau)}}{\alpha} - \frac{e^{-\alpha(T-\tau)} - e^{-2\alpha(T-\tau)}}{\alpha}\right) \\
&- \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa(\alpha-\kappa)} - \frac{e^{\kappa(T-\tau)} - e^{-(\alpha-\kappa)(T-\tau)}}{\alpha(\alpha-\kappa)} - \frac{1}{\alpha}\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right)\right) \\
&- \frac{1}{2}\sigma_S^2(T-\tau) - \frac{\rho_{12}\sigma_S\sigma_r}{\alpha}\left(T-\tau - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right) - \frac{\rho_{13}\sigma_S\psi}{\kappa}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa} - T + \tau\right), \\
\mu_{r_T|r_\tau} =& e^{-\alpha(T-\tau)} r_\tau + \left(\bar{\gamma} - \frac{\sigma_r^2}{\alpha^2}\right)\left(1-e^{-\alpha(T-\tau)}\right) + \frac{\sigma_r^2}{2\alpha^2}\left(1-e^{-2\alpha(T-\tau)}\right) \\
&- \frac{\rho_{23}\sigma_r\psi}{\kappa}\left(\frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa} - \frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right), \\
\mu_{\mu_{x+T}|\mu_{x+\tau}} =& \mu_{x+\tau} e^{\kappa(T-\tau)} - \frac{\rho_{23}\sigma_r\psi}{\alpha}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa} - \frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa}\right) \\
&- \frac{\psi^2}{\kappa}\left(\frac{e^{2\kappa(T-\tau)}-1}{2\kappa} - \frac{e^{\kappa(T-\tau)}-1}{\kappa}\right),
\end{aligned}$$

$$
\begin{aligned}
\sigma^2_{q_T|q_\tau} =& \sigma^2_S(T-\tau) + \frac{\sigma^2_r}{\alpha^2}\left(T-\tau-2\frac{1-e^{-\alpha(T-\tau)}}{\alpha}+\frac{1-e^{-2\alpha(T-\tau)}}{2\alpha}\right)\\
&+\frac{2\rho_{12}\sigma_S\sigma_r}{\alpha}\left(T-\tau-\frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right),\\
\sigma_{q_T,r_T|q_\tau,r_\tau} =& \rho_{12}\sigma_S\sigma_r\left(\frac{1-e^{-\alpha(T-\tau)}}{\alpha}\right)+\frac{\sigma^2_r}{\alpha}\left(\frac{1-2e^{-\alpha(T-\tau)}+e^{-2\alpha(T-\tau)}}{2\alpha}\right),\\
\sigma_{q_T,\mu_{x+T}|q_\tau,\mu_{x+\tau}} =& \rho_{13}\sigma_S\psi\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa}\right)+\frac{\rho_{23}\sigma_r\psi}{\alpha}\left(\frac{e^{\kappa(T-\tau)}-1}{\kappa}-\frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa}\right),\\
\sigma^2_{r_T|r_\tau} =& \sigma^2_r\left(\frac{1-e^{-2\alpha(T-\tau)}}{2\alpha}\right),\\
\sigma_{r_T,\mu_{x+T}|r_\tau,\mu_{x+\tau}} =& \rho_{23}\sigma_r\psi\left(\frac{1-e^{-(\alpha-\kappa)(T-\tau)}}{\alpha-\kappa}\right),\\
\sigma^2_{\mu_{x+T}|\mu_{x+\tau}} =& \psi^2\left(\frac{e^{2\kappa(T-\tau)}-1}{2\kappa}\right).
\end{aligned}
$$

It is possible to write the conditional mean of $Y_T$ given $Y_\tau$ in the following affine form:

$$
\begin{aligned}
\begin{bmatrix}\mu_{q_T|q_\tau}\\ \mu_{r_T|r_\tau}\\ \mu_{\mu_{x+T}|\mu_{x+\tau}}\end{bmatrix} =& \underbrace{\begin{bmatrix}1 & \frac{1-e^{-\alpha(T-\tau)}}{\alpha} & 0\\ 0 & e^{-\alpha(T-\tau)} & 0\\ 0 & 0 & e^{\kappa(T-\tau)}\end{bmatrix}}_{H}\begin{bmatrix}q_\tau\\ r_\tau\\ \mu_{x+\tau}\end{bmatrix}+C_\tau\\
=& HY_\tau + C_\tau,
\end{aligned}
$$

where $C_\tau$ is a constant matrix defined by remaining terms of mean vector of $Y_T|Y_\tau$ after defining $HY_\tau$. The unconditional distribution of $Y_T$ under $\tilde{\mathbb{P}}$ is also Gaussian since $Y_\tau$ and $Y_T|Y_\tau$ follow Gaussian distributions. Thus, it suffices to specify a mean vector and a covariance matrix of $Y_T$ under $\tilde{\mathbb{P}}$ to specify its distribution:

$$
\begin{aligned}
\mu_T = \mathbb{E}^{\tilde{\mathbb{P}}}[Y_T] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}_E}[Y_T|Y_\tau]\right] = \mathbb{E}^{\mathbb{P}}\left[HY_\tau + C_\tau\right] = H\mu_\tau + C_\tau,\\
\Sigma_T = \mathrm{Cov}^{\tilde{\mathbb{P}}}[Y_T] &= \mathrm{Cov}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}_E}[Y_T|Y_\tau]\right] + \mathbb{E}^{\mathbb{P}}\left[\mathrm{Cov}^{\mathbb{Q}_E}[Y_T|Y_\tau]\right]\\
&= \mathrm{Cov}^{\mathbb{P}}\left[HY_\tau + C_\tau\right] + \mathbb{E}^{\mathbb{P}}\left[\Sigma_{T|\tau}\right] = H\Sigma_\tau H' + \Sigma_{T|\tau}.
\end{aligned}
$$

Hence, $Y_T \sim N(\mu_T,\ \Sigma_T)$.

The final step is to specify the joint distribution of $Y_\tau$ and $Y_T$ by finding $\mathrm{Cov}(Y_\tau, Y_T)$. Note that:

$$
\begin{aligned}
\Gamma = \mathrm{Cov}(Y_\tau, Y_T) &= \mathbb{E}^{\tilde{\mathbb{P}}}[Y_\tau Y_T'] - \mathbb{E}^{\tilde{\mathbb{P}}}[Y_\tau]\mathbb{E}^{\tilde{\mathbb{P}}}[Y_T']\\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{Q}_E}[Y_\tau Y_T'|Y_\tau]] - \mu_\tau\mu_T'\\
&= \mathbb{E}^{\mathbb{P}}\left[Y_\tau(Y_\tau'H' + C_\tau')\right] - \mu_\tau\mu_T'\\
&= \Sigma_\tau H'.
\end{aligned}
$$

Therefore,

$$
\begin{bmatrix}Y_\tau\\ Y_T\end{bmatrix} \sim N\left(\begin{bmatrix}\mu_\tau\\ \mu_T\end{bmatrix},\quad \begin{bmatrix}\Sigma_\tau & \Gamma\\ \Gamma' & \Sigma_T\end{bmatrix}\right).
$$

□

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