The Valuation of Liabilities, Economic Capital, and RAROC in a Dynamic Model∗

Daniel Bauer & George Zanjani†

Department of Economics, Finance, and Legal Studies. University of Alabama
361 Stadium Drive, Tuscaloosa, AL 35487. USA

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Abstract

We develop an economic capital model of an insurer operating in a dynamic setting. The dynamic results suggest two important modifications to solvency assessment and performance measurement via risk-adjusted return ratios, both of which are typically rooted in a static approach. First, “capital” should be defined broadly to include the continuation value of the firm. Second, cash flow valuations must reflect risk adjustments to account for company effective risk aversion. We illustrate our results using data from a catastrophe reinsurer, finding that the dynamic modifications are practically significant—although static approximations with a properly calibrated company risk aversion are quite accurate.

JEL classification: G22; G32; C63

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†Corresponding author. Phone: +1-(205)-348-6291. Fax: +1-(205)-348-0590. E-mail addresses: dbauer@cba.ua.edu (D. Bauer); ghzanjani@cba.ua.edu (G. Zanjani).
1 Introduction

Economic Capital (EC) models are increasingly important both in the context of solvency regulation in insurance (Solvency II, Swiss Solvency Test (SST)) and in internal steering. EC models are typically motivated in a static setting, where the portfolio risks are evaluated through application of a risk measure such as Value-at-Risk (VaR) at a given risk horizon. This approach sets up answers to two critically important questions. First, by determining how much capital is required to keep the risk below a tolerance level, EC models address the question of how much capital should be held. Second, by calculating the gradient of the risk measure, EC models address the question of how much capital is needed to support a particular exposure and, hence, what the exposure costs the company. The latter calculation underlies RAROC and similar performance evaluation techniques, where the ratio of an exposure’s expected return to allocated supporting capital is compared to a hurdle rate for evaluating its profitability.

Unfortunately, there is a mismatch between the underlying model and reality. Financial institutions are not static in nature. They operate as going concerns in a dynamic environment, with underwriting decisions interacting with other external financing decisions in real time. It has long been understood (Froot and Stein [1998]) that value maximization in dynamic settings leads to risk pricing results that are incongruent with those produced by static approaches. Yet the course of scholarship, as well as practice, has continued to develop within the static model paradigm, and the problem of reconciling capitalization and pricing guidance from this paradigm with the complications of dynamic contexts continues to fester.

We revisit this problem of reconciliation. We study a dynamic model of an insurer, allowing for varied opportunities to raise financing from customers as well as investors. We find that risk pricing and solvency assessment in this model can be reconciled with the static approach—although only if one relies on adequate notions of capitalization, the hurdle rate, and the expected return of exposures to account for the dynamic nature of the problem.

First, “capital” must be defined differently from static approaches, which typically conceive capital as something akin to the current book equity of the firm or the market-consistent value of the existing portfolio. In our dynamic model, the cash capital on hand is not the only resource

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1 Solvency II is a directive within the European Union that codifies and harmonizes insurance regulation and that came into effect in January of 2016, although there are still transition rules in place. In contrast to former insurance regulatory frameworks, Solvency II is risk-based and explicitly allows—and to some extent encourages—companies to rely on enterprise-wide internal economic capital models, an option that has been taken by most of the large insurance carriers. See EIOPA (2012) for details. The SST is a similar framework in Switzerland, see SFINMA (2006).

2 Return on Risk-Adjusted Capital (RORAC) and Risk-Adjusted Return on Risk-Adjusted Capital (RARORAC) are also discussed, with devotees of the latter in particular arguing for the importance of risk adjustments both in the numerator and denominator. Our sense, however, is that practical distinctions among these ratios are not universally agreed upon. Absent a definitive nomenclature, we utilize “RAROC,” the most widely used term of the set, as a generic term for a return on capital measure that has been adjusted for risk in some sense.
protecting the firm. The relevant notion of capital includes untapped resources that might be accessed in an emergency. These untapped resources are the amounts that can be raised in the event of financial distress and equate to the value of the firm as a going concern in the event of financial impairment—i.e., the maximum amount the company is worth should the financial assets on hand be insufficient to meet its obligations. This expanded notion of capital is obviously relevant for solvency assessment, as insolvency will only happen in situations where the firm’s obligations are so great that the firm is not worth saving. It is also the right concept for risk pricing: Our dynamic version of RAROC allocates “capital” to the various risks in the portfolio, but the capital allocated includes all assets, including these untapped resources.

Second, the hurdle rate, which is typically a target return on equity in the static model, also requires adjustment. At one level, this is not surprising: since the definition of capital has changed, so has the conception of its cost. In the dynamic approach, the appropriate hurdle rate can still be interpreted as a marginal cost of raising capital, but the cost of the marginal unit of capital must be expressed net of its contribution to the continuation value of the firm.

Finally, the expected return of the exposure must be adjusted to account for the firm’s effective risk aversion. It is well-known that external financing frictions can produce risk avverting behavior of a value-maximizing firm. We show that this effective risk aversion can be reflected through an endogenously determined weighting function that weights outcomes according to their impact on the firm’s value. This impact differs from the usual market-consistent effects due to the presence of frictions, which generate firm-specific value influences.

After developing the theoretical results in Section 2, we explore their practical significance in Section 3. We implement a calibrated version of the model with numerical techniques, using simulated data provided by a catastrophe reinsurer for the liability portfolio. We solve for dynamically optimal underwriting and financing decisions, and then compare the dynamic RAROCs from the solved model to their static counterparts.

We show that failure to account for the modifications discussed above leads to significant distortions in the assessed profitability of the various lines of insurance. For example, overall levels of line profitability are significantly overstated when using a narrow definition of capital (rather than a broad one including the continuation value of the firm), and relative profitability across lines is significantly distorted when failing to make adjustments for the firm’s effective risk aversion. Both effects are especially pronounced for undercapitalized firms.

A potential criticism of the dynamic approach is its computational complexity, as the approach requires a complete specification and solution of the firm’s dynamic problem. To address this, we also explore the accuracy of approximations to the dynamic results using static models where the

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3This theoretical result has practical significance: troubled banks and insurance companies are often rescued by competitors either through marketplace transactions or through marriages arranged by regulators.
firm’s effective risk aversion is captured through a simple CRRA utility function. We show that RAROCs calculated under this approach, when using a properly calibrated risk aversion coefficient (roughly 0.2 in our application) and the appropriate broad definition of capital, provide very close approximations to the fully specified and solved dynamic RAROCs.

**Relationship to the Literature and Organization of the Paper**

Despite the increasing importance of EC models for financial institutions, there is no “global consensus as to how to define and calculate” EC (Society of Actuaries, 2008). For instance, (i) the form of the penalty for non-marketed (firm-specific) risk, (ii) the cost of capital figure, or (iii) the requirement of how much capital to hold (e.g., what to include in the notion of capital or what risk measure to use) are subject to debate. There are numerous papers in specialized and practitioner-oriented literatures highlighting or weighing in on these debates (see e.g. Pelsser and Stadje (2014) and Engsner, Lindholm, and Lindskog (2017) on (i); Tsanakas, Wüthrich, and Cerny (2013) on (ii); or Danielsson et al. (2001), Embrechts et al. (2014), and Burkhart, Reuß, and Zwiesler (2015) on (iii)). By incorporating multiple periods with different modes of capitalization in an economic framework with financial frictions in the spirit of Froot and Stein (1998), our model delivers marginal equations in direct analogy to the EC frameworks applied in practice—allowing us to weigh in on some of these debates. In particular, our framework clarifies the notion of total capital relevant for internal steering and suggests that future capital costs should be captured by a firm-specific weighting affecting valuation (rather than recursive modifications in the risk margin term as conceived in regulatory frameworks, e.g. Möhr (2011)).

Our results also have implications for the application of EC Models for internal steering, particularly for performance measurement via RAROC. Theoretical foundations for such ratios, as well as their component pieces of marginal return and allocated capital, are easily established in the context of single-period optimization models (see e.g. Tasche (2000), Gourieroux, Laurent and Scailet (2000), Denault (2001), Myers and Read (2001), Zanjani (2002), Kalkbrener (2005), Stoughton and Zechner (2007), or Bauer and Zanjani (2016) for earlier research on capital allocation and RAROC). The connection between RAROC with the optimality conditions emerging from multi-period models is less well understood. Froot and Stein (1998) applied the model of Froot, Scharfstein, and Stein (1993) to the context of financial institutions facing costly external financing, concluding that hurdle rates should be adjusted to account for institution-specific risk aversion. They expressed the hurdle rate for opportunities as a two-factor pricing model (later generalized...

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4 We do not model the equilibrium origin of these frictions but take them as exogenous. We refer to the growing literature on macro-economic frictions (Duffie 2010b, Gromb and Vayanos 2010, Brunnermeier, Eisenbach, and Sannikov 2013, e.g.) and particularly Appendix D of Duffie and Strulovici 2012 that presents a version of their equilibrium model with capital mobility frictions tailored to catastrophe insurance corresponding to our numerical application.
to a three-factor model for insurance companies by Froot (2007)) and explored reconciliation with RAROC, ultimately finding no intuitive connection, as well as a number of practical difficulties in implementation that lead RAROC to be inconsistent with value maximization. This skepticism is echoed by Erel, Myers, and Read (2015), who do not find a closed form expression to reconcile RAROC with their value-maximizing calculations. In our model, we are able to reinterpret the marginal cost of risk within the framework of RAROC. This reinterpretation does require a significant investment in solving the optimization problem of the financial institution, echoing Froot and Stein’s observation that there are significant practical difficulties in implementing a “correct” approach. Our reinterpretation, however, allows us to compare the results of a corrected RAROC approach with the more typical implementations of RAROC, and we do this using real-world data for the case of a catastrophe reinsurer.

2 Economic Capital and the Marginal Cost of Risk

The typical portfolio optimization model considers the maximization of profits subject to a risk measure constraint in a single period. In an insurance setting, the marginal cost of risk consists of two parts—a marginal actuarial cost and a risk charge that can be interpreted as a capital allocation times a cost of capital. The RAROC ratio is then calculated by deducting the marginal actuarial cost from the price and dividing by the allocated capital, a ratio which is then compared with the cost of capital or hurdle rate for the exposure (see Appendix A.2 for a summary).

In this section, we reconsider the connection between the marginal cost of risk and the allocation of (risk) capital in a more general dynamic setting for an insurance company. The cost of risk reflects two important influences. First, less risky insurers are able to charge higher prices for insurance coverage due to risk aversion of their customers. Second, greater risk produces a higher probability of financial distress, which brings the burden of costly external financing and potential default. In the event of default, the owners lose their claim to future profit flows. These two influences create effective risk aversion at the level of the company and motivate holding of capital despite its carrying cost.

As detailed in Section 2.1, the company maximizes (risk-neutral) value by choosing its participation in covering various risks, and its capital raising and shedding (dividend) decisions. The optimization problem yields a Bellman equation, where firm value is a function of the current capital level. At any point in time, the firm may be over- or under-capitalized: Too little capital in the firm leads the company to forego profitable business opportunities, whereas too much capital is too costly relative to (decreasing) profit margins. This is also reflected in the optimal capital raising

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\textsuperscript{5}Erel, Myers, and Read (2015) do not explicitly analyze multiple periods, but they do incorporate similar effects in reduced form cost functions intended to reflect costs of financial distress.
decision: A meagerly capitalized company will raise funds whereas an over-capitalized firm will shed by paying dividends. In case the company is underwater after losses are realized, the company will be bailed out at (high) emergency raising costs if doing so is economical.

We go on in Section 2.2 to derive the marginal cost of each risk in the company’s portfolio, from which the RAROC of the risk can be generated. A RAROC ratio similar to that obtained from a static one-period model can be recovered in our analysis of the dynamic model, with modifications to the denominator, the hurdle rate, and the numerator as outlined in the Introduction. In particular, it is worth noting that the resulting marginal cost aligns exactly with that from a simple one-period model if the company were endowed with a suitable utility function and cost of capital (again see Appendix A.2 for details)—with the key difference that we are able to derive an endogenous expression of the company’s effective utility function due to the financing frictions.

2.1 Profit Maximization Problem in a Multi-Period Model

Formally we consider an insurance company with $N$ business lines and corresponding loss realizations $L_t^{(i)}$, $i = 1, 2, \ldots, N$, each period $t = 1, 2, \ldots$ These losses could be associated with certain perils, certain portfolios of contracts, or even individual contracts/customers.

We assume that for fixed $i$, $L_1^{(i)}, L_2^{(i)}, \ldots$ are non-negative, independent, and identically distributed (iid) random variables. We make the iid assumption for convenience of exposition, and since it suits our application in Section 3. However, non-identical distributions arising from, e.g., claims inflation could be easily incorporated, and also extensions to serially correlated (e.g., autoregressive) loss structures or loss payments developing over several years are feasible at the expense of a larger state space.

We also abstract from risky investments, so that all the uncertainty is captured by the losses; we define the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ that describes the information flow over time via $\mathcal{F}_t = \sigma(L_s^{(i)}, i \in \{1, 2, \ldots, N\}, s \leq t)$. However, generalizations with securities markets are possible at the expense of notational complication.

At the beginning of every underwriting period $t$, the insurer chooses to underwrite certain portions of these risks and charges premiums $p_t^{(i)}$, $1 \leq i \leq N$, in return. More precisely, the underwriting decision corresponds to choosing an indemnity parameter $q_t^{(i)}$, so that the indemnity for loss $i$ in period $t$ is:

$$I_t^{(i)} = I_t^{(i)}(L_t^{(i)}, q_t^{(i)}),$$

where we require $I_t^{(i)}(0, q_t^{(i)}) = 0, i = 1, 2, \ldots, N$. For analytical convenience and again because it suits our setting in Section 3, we focus on choosing to underwrite a fraction of the risks, i.e., we assume:

$$I_t^{(i)} = I^{(i)}(L_t^{(i)}, q_t^{(i)}) = q_t^{(i)} \times L_t^{(i)},$$
although, here also, generalizations are possible. We denote the aggregate period-\(t\) loss by \(I_t = \sum_i I_t^{(i)}\).

We consider an environment with financing frictions [Duffie, 2010b; Gromb and Vayanos, 2010; Brunnermeier, Eisenbach, and Sannikov, 2013, e.g.], although we do not explicitly model their equilibrium origin.\(^6\) Thus, there is a cost associated with carrying and raising capital, where our assumptions reflect that “external funding is [...] more expensive than internal funding through retained earnings” (Brunnermeier, Eisenbach, and Sannikov, 2013). Specifically, we assume the company has the possibility to raise or shed (i.e., pay dividends) capital \(R^b_t\) at the beginning of the period at cost \(c_1(R^b_t)\), \(c_1(x) = 0\) for \(x \leq 0\), and that there exists a positive carrying cost for capital \(a_t\) within the company as a proportion \(\tau\) of \(a_t\), where \(c'_1(x) > \tau, x > 0\).

In addition, we allow the company to raise capital \(R^e_t\), \(R^e_t \geq 0\), at the end of the period—at a (higher) cost \(c_2(R^e_t)\). Here we think of \(R^b_t\) as capital raised under normal conditions, whereas \(R^e_t\) is emergency capital raised under distressed conditions.\(^7\) In particular, we assume:

\[
c'_2(x) > c'_1(y), \quad x, y > 0,
\]

i.e., raising a marginal dollar of capital under normal conditions is less costly than in distressed states.

Finally, the (constant) continuously compounded risk-free interest rate is denoted by \(r\). Hence, the law of motion for the company’s capital (budget constraint) is:

\[
a_t = \left[ a_{t-1} \times (1 - \tau) + R^b_t - c_1(R^b_t) + \sum_{j=1}^{N} p_t^{(j)} \right] e^r + R^e_t - c_2(R^e_t) - \sum_{j=1}^{N} I_t^{(j)}
\]

for \(a_{t-1} \geq 0\). We require that:

\[
R^b_t \geq -a_{t-1}(1 - \tau),
\]

i.e., the company cannot pay more in dividends than its capital (after capital costs have been deducted).

The company defaults if \(a_t < 0\), which is equivalent to:

\[
\left[ a_{t-1} \times (1 - \tau) + R^b_t - c_1(R^b_t) + \sum_{j=1}^{N} p_t^{(j)} \right] e^r + R^e_t - c_2(R^e_t) < \sum_{j=1}^{N} I_t^{(j)}.
\]

Due to limited liability, in this case the company’s funds are not sufficient to pay all the claims.

\(^6\)See e.g. Appendix D of Duffie and Strulovici [2012], where the authors present a version of their equilibrium model with capital mobility frictions that is tailored to catastrophe insurance.

\(^7\)Warren Buffett’s investments in Swiss Re and Goldman Sachs during the financial crisis provide examples of the high cost of financing under conditions of distress in insurance and banking, respectively.
We assume that the remaining assets in the firm are paid to claimants at the same rate per dollar of coverage, so that the recovery for policyholder $i$ is:

$$\min \left\{ I^{(i)}_t, \frac{D_t}{\sum_{j=1}^N f^{(j)}_t} \times I^{(i)}_t \right\},$$

where:

$$D_t = \left[ a_{t-1} \times (1 - \tau) + R^b_t - c_1(R^b_t) + \sum_{j=1}^N p^{(j)}_t \right] e^r + R^e_t - c_2(R^e_t)$$

are the financial resources the company has available to service indemnities.

The premium the company is able to charge for providing insurance now depends on the riskiness of the coverage as well as the underwriting decision—that is, price is a function of demand as within an *inverse demand function*. Formally, this means that the total premium for line $i$, $p^{(i)}_t$, is a quantity known at time $t-1$ (i.e., it is $\mathcal{F}$-predictable) given by a functional relationship:

$$p^{(i)}_t \left( a_{t-1}, R^b_t, R^e_t, (p^{(j)}_t)_{1 \leq j \leq N}, (q^{(j)}_t)_{1 \leq j \leq N} \right) = 0, \ 1 \leq i \leq N.$$  

We use a reduced-form specification that assumes premiums—as markups on discounted expected losses—are a function of the company’s aggregate loss $\mathbb{E}[I]$ and a *risk functional* $\phi(I_t, D_t)$ that measures risk as a function of the aggregate indemnity random variable and the total resources available to the company:

$$p^{(i)}_t = e^{-r} \mathbb{E}_{t-1} [I^{(i)}_t] \exp \left\{ \alpha - \gamma \mathbb{E}_{t-1} [I] - \beta \phi(I_t, D_t) \right\}. \quad (4)$$

Note that since the premiums appear in $D_t$, the constraint is still implicit. Aside from natural monotonicity assumptions ($\phi(I, x) \leq \phi(I, y)$, $x \geq y$, and $\phi(I, x) \leq \phi(J, x)$ for $I \leq J$ a.s.), the primary assumption is that the risk functional is scale invariant: $\phi(aI, ax) = \phi(I, x)$, $a \geq 0$. The key example that we have in mind is the company’s default probability, $\phi(I_t, D_t) = \mathbb{P}_{t-1}(I_t > D_t)$, with the intuition that consumers rely on insurance solvency ratings for making their decisions. We will rely on this specification in the main text for the ease of presentation and in our application in the next section, so that we set:

$$p^{(i)}_t = e^{-r} \mathbb{E}_{t-1} [I^{(i)}_t] \exp \left\{ \alpha - \gamma \mathbb{E}_{t-1} [I] - \beta \mathbb{P}_{t-1}(I_t > D_t) \right\}. \quad (5)$$

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8 Alternative bankruptcy rules may be used without affecting the results, with the only caveat that all remaining assets must be paid out to policyholders.
However, for the technical presentation in the Appendix we will rely on the general form. Obviously, we expect both $\beta$ and $\gamma$ to have a positive sign, i.e., the larger the default rate the smaller the premium loading and the more business the company writes the smaller are the profit margins, respectively. Of course, other generalizations such as line-specific parameters are straightforward to include in theory, but they will complicate the estimation as well as the (numerical) solution of the optimization problem.

We assume that the company is risk-neutral and maximizes expected profits net of financing costs as described above, so that it solves:

$$V(a) = \max_{\{p_t^{(j)}\}, \{q_t^{(j)}\}, \{R_t^p\}, \{R_t^e\}} \left\{ \mathbb{E} \left[ \sum_{t=1}^{\infty} I_{\{a_1 \geq 0, \ldots, a_t \geq 0\}} e^{-rt} \left[ e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} \right] \right. \right.$$

$$\left. - (r a_{t-1} + c_1(R_t^b)) e^r - c_2(R_t^e) \right] - I_{\{a_1 \geq 0, \ldots, a_{t-1} \geq 0, a_t < 0\}} e^{-rt} \left[ (a_{t-1} + R_t^b) e^r + R_t^e \right] \right\},$$

subject to $2)$; $3)$; $5)$; $R_t^e \geq 0$; $\{p_t^{(j)}\}, \{q_t^{(j)}\}, \{R_t^p\}$ $\mathbf{F}$-predictable; and $\{R_t^e\}$ $\mathbf{F}$-adapted.\footnote{Note that the expectation may entail risk-adjusted or risk-neutral probabilities for marketed risks (this will be especially relevant in a setting with securities markets). In our application in the next Section, we evaluate insurance risks under physical probabilities, which is in line with industry practice for models with capital costs and may be justified by the assumption that the market for insurance risk is “small” relative to financial markets (Bauer, Phillips, and Zanjani, 2013).}

In the Appendix (Section A.1), we show that the objective function above can be expressed, after rearrangement, as the discounted present value of future dividends:

$$V(a) = \max_{\{p_t^{(j)}\}, \{q_t^{(j)}\}, \{R_t^p\}, \{R_t^e\}} \left\{ \mathbb{E} \left[ \sum_{t^* \geq 0} I_{\{a_1 \geq 0, a_2 \geq 0, \ldots, a_{t^*-1} \geq 0, a_{t^*} < 0\}} e^{-rt} \left[ -e^r R_t^b - R_t^e \right] \right] - a_0 \right\},$$

We go on in the Appendix to use the principles of dynamic programming to recast the problem as a Bellman equation:

$$V(a) = \max_{\{p^{(j)}\}, \{q^{(j)}\}, R^p, R^e} \left\{ \mathbb{E} \left[ I_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)} e^r + R^e - c_2(R^e) \geq I)\}^* \right] \times \right. \right.$$

$$\left. \left( \sum_j p^{(j)} e^{-r I} - e^{-r} I - \tau a - c_1(R^b) - e^{-r} c_2(R^e) \right) + e^{-r} V \left( [a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} e^r + R^e - c_2(R^e) - I] \right) \right. \right.$$

$$\left. - I_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)} e^r + R^e - c_2(R^e) < I\}} \left( a + R^b + e^{-r} R^e \right) \right\},$$

$$\left. \right\}$$

\footnote{We omit other potentially relevant constraints, e.g. arising from regulatory capital requirements. These could be easily incorporated without changing the key conclusions.}
subject to (3) and (5).

Characterizing the optimal financing policy under (8), it is clear that the optimal choice for emergency capital raising, \( R^e \), follows a simple rule. Since emergency capital raising is always more expensive than raising capital under normal conditions at the margin (see Equation (11)), the amount raised will either be zero—in cases where the company is solvent or so far under water that it is not worth saving—or exactly the amount required to save the company by paying all of its unmet obligations. We can thus identify three key decision regions for the insurer based on the total claims submitted, with the key thresholds being

$$ S = \left[ a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r, $$

the total assets held by the insurer before claims are received, and \( D \), the default threshold—

with \( D > S \):

1. \( I \leq S \) : Claims are less than the assets held by the insurer. No emergency raising is necessary: \( R^e = 0 \).

2. \( S < I \leq D \) : Claims are greater than the assets held by the insurer but less than the threshold at which it is optimal to default: \( R^e - c_2(R^e) = I - S \).

3. \( I > D \) : Claims are greater than the default threshold. The company does not have sufficient assets to pay claims, and the shortfall is too great to justify raising money to save the company: \( R^e = 0 \).

The default threshold equates the cost of saving the company and the value of an empty company (see Appendix A.1 for a formal statement):

$$ V(0) = R^e = (D - S) + c_2(R^e) \iff D = S + [V(0) - c_2(V(0))]. $$

Armed with this insight, we specialize further to the case where the cost of raising emergency capital is linear in the amount raised:

$$ c_2(R^e) = \xi R^e. $$

We can then rewrite the Bellman equation as:

$$ V(a) = \max_{\{p^{(j)}\}, \{q^{(j)}\}, R^b} \left\{ \mathbb{E} \left[ e^{-r} \left( I_{\{I \leq S\}} \times (\lfloor S - I \rfloor + V(S - I)) + I_{\{S < I \leq D\}} \times (V(0) - \frac{I - S}{1 - \xi}) \right) \right] - [a + R^b] \right\} \tag{9} $$
subject to (3) and

\[ p^{(i)} = e^{-r} \mathbb{E} \left[ I^{(i)} \right] \times \exp \left\{ \alpha - \gamma \mathbb{E}[I] - \beta \mathbb{P}(I > D) \right\}. \] (10)

### 2.2 The Marginal Cost of Risk and RAROC

In what follows, we continue to assume a linear cost for end-of-period capital, so that we study problem (9) subject to the constraints (3) and (10). As shown in the Appendix, we work with optimality conditions to obtain an expression for the balancing of marginal revenue with marginal cost for the \( i \)-th risk:

\[
\text{MR}_i = \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} w(I) I_{\{I \leq S\}} \right] + \frac{\partial}{\partial q_i} \text{VaR}_\psi(I) \times \mathbb{E} \left[ w(I) I_{\{I > D\}} \right],
\] (11)

where \( \psi = \mathbb{P}(I > D) \),

\[
\mathbb{E}[w(I)] = 1 \text{ with } w(I) = \begin{cases} 
(1 - c'_{1}) \times (1 + V'(S - I)) & , I \leq S, \\
(1 - c'_{1}) \times \frac{1}{1 - \xi} & , S < I \leq D, \\
e^{r \frac{f_I(D)}{\mathbb{P}(I > D)}} \beta \sum_j p^{(j)} & , I > D,
\end{cases}
\] (12)

and \( f_I \) denotes the probability density of \( I \).

To appreciate the significance of this result, it is useful to consider a one-period model that leads to RAROC calculations prevalent in practice, where the company chooses an optimal portfolio and costly capital (assets) \( S \). The problem is framed in different ways, where one common formulation maximizes profits subject to a risk measure constraint ([Tasche, 2000][McNeil et al., 2005]). In our setting, the risk measure emanates from the consumers’ perception of risk, and [Zanjani (2002)] shows these formulations yield equivalent results. Since consumers evaluate the company via the default probability in (5), it is no surprise that Value-at-Risk (VaR) emerges—since it is the risk measure that has “its focus on the probability of a loss regardless of the magnitude” ([Basak and Shapiro, 2001]).

Appendix A.2 (Eq. (19)) shows that we obtain the following expression for the marginal rev-
 enqueue in such a one-period version of our model:

\[
\text{MR}_i = \mathbb{E} \left[ \frac{\partial I(i)}{\partial q(i)} \right] \exp \left\{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \right\} (1 - \gamma \mathbb{E}[I]) \tag{13}
\]

Here \( \text{MR}_i \) presents the marginal revenue associated with an increase in the exposure to the \( i \)-th risk keeping the company risk level constant. Since the optimality conditions balance revenues and costs at the optimum, \( \text{MR}_i \) will also equal the **marginal cost of risk**. And in this simple model, the marginal cost of risk \( i \) consists of (I) the marginal increase in indemnity payments in solvent states plus (II) marginal capital costs allocated to the portfolio risks according to the gradient of the risk measure. Here, capital costs consist of the direct cost associated with raising capital (assets) \( S \) plus the default probability—since the marginal dollar of capital will be lost in default states.

A different representation of (13) is the Risk-Adjusted Return On Capital (RAROC), stating that the marginal return over allocated capital for each risk should equal exactly the capital costs:

\[
\text{RAROC}_i = \frac{\text{MR}_i - \mathbb{E} \left[ \frac{\partial I(i)}{\partial q(i)} 1_{I \leq S} \right]}{\partial q_i \text{VaR}_{\mathbb{P}(I > S)}(I)} = \left[ c'_i + \mathbb{P}(I > S) \right].
\]

Practical applications evaluate the RAROC for each line relative to the hurdle rate \( c'_i + \mathbb{P}(I > S) \) for the purposes of pricing and performance measurement \cite{McKinsey&Company, 2011, Society of Actuaries, 2008}.

Comparing expression (13) for the simple one-period model and expression (11) for our multi-period model with different modes of capitalization, the general form of the marginal cost of risk remains the same but there are three differences worth noting.

First, in the simple setting, the company does not have access to end-of-period capital raising so that the relevant cutoff is the chosen asset level \( S \). The multi-period setting entails a broader notion of capital that considers all resources, including end-of-period capital raising. While this originates from the way the problem is set up (where policyholders worry about the default threshold \( D \)), it is interesting to note that:

\[
D - S = (1 - \xi) V(0),
\]

which, according to (7), is the cost-adjusted present value of future profits (PVFP) for a zero capital firm. Thus, the relevant notion of capital in our setting includes the discounted franchise value, where the discount rate corresponds to the cost of capital in financial distress. Our results suggest that capital should be defined on a **going concern basis**, since firm value can be pledged.
to avoid insolvency, echoing arguments from the specialized insurance literature in the context of Solvency II capital definition (Burkhart, Reuß, and Zwiesler, 2015 and references therein).

Second, the capital cost $c'$ does not directly enter the “hurdle rate” $E \left[ w(I) 1_{\{I > D\}} \right]$ but is included in the weighting function $w$. The reason is that while in the one-period model, capital costs are directly assessed for raising assets $S$, in the multi-period model this year’s premium does not have to directly provide for $[c' D]$, as there are a variety of interacting capitalization options. Rather, upon default the company loses access to capital $D$, the value of which is assessed by a risk-adjusted default probability. Thus, the hurdle rate in this context has a precise interpretation and is not an “arbitrary, exogenously specified constant figure” as in practical solvency frameworks (Tsanakas, Wüthrich, and Cerny, 2013).

This leads us to the third difference: The expression in the multi-period model (11) entails a weighting $w$ of different aggregate loss states $I$ with $E[w(I)] = 1$. To interpret this weighting function, it is again helpful to turn to the basic one-period model but under the assumption that the company is risk-averse and evaluates future cash flows via a (given) utility function $U$. The second part of Appendix A.2 shows that in this situation of a risk-averse insurer we obtain (cf. Eq. (20) in App. A.2):

$$MR_i = E \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \tilde{w}(I) 1_{\{I \leq S\}} + \frac{\partial}{\partial q_i} \text{VaR}_{P(\tilde{I} > S)}(I) \times \left[ c'_1 + E \left[ \tilde{w}(I) 1_{\{I > S\}} \right] \right] \right],$$

(14)

where $\tilde{w} = U' / E[U']$ so that $E[\tilde{w}(I)] = 1$.

We again note the similarity between the marginal cost of risk in the simple one-period setting (14) and its counterpart in the multi-period model (11). Both entail a weighting function, and—for the simple model—it’s origin is straightforward: Payments by the company are not valued according to their actuarial cost (expected present values). Instead, state-weights associated with the company’s preferences enter the valuation. This is familiar from conventional micro-economic and asset pricing theory, where cash flows are weighted using state price densities or stochastic discount factors (Duffie, 2010a, e.g.). In this case, the market value weights (which correspond to actuarial probabilities in our simplified case) are adjusted for the risk aversion of the institution.

The interpretation in the multi-period version is analogous. A marginal increase in risk $i$ will produce changes in end-of-period outcomes, which will affect the value of the company—that is, there is a (random) cost associated with the continuation value of the company. In solvent states ($I \leq S$) the relevant cost will be the marginal company valuation $V'(S - I)$ whereas in distressed states ($S < I \leq D$) the increase in exposure leads to an increase in the expected costs associated with saving the company at cost $\xi$. The factor $(1 - c'_1)$ reflects the fact that in the multi-period model, premiums act as a substitute for capital raised and thus save the company the marginal cost of raising capital. The weight in default states, similarly, follows from the value
the company places on a marginal dollar in default states. Reducing the default probability will increase the premium income \( e^{r} \sum_j p^{(j)} \), where the sensitivity of premiums to the default rate is given by \( \beta \) and \( f_{I(D)}/P(I>D) \) is the relative sensitivity at the default threshold relative to capital in the tail. Importantly, the weighting function integrates to one because of the parity constraint at the optimum. The marginal value of raising one extra dollar to the company is exactly one dollar, otherwise the company would raise more. In particular, the “hurdle rate” \( \mathbb{E}[w(I) 1_{\{I>D\}}] \) can be derived from the valuation weights in solvent states.

This weighting thus reflects a central insight from the theoretical literature on risk management in the presence of financing frictions, namely that financing constraints render financial institutions effectively risk averse (Froot, Scharfstein, and Stein [1993]; Froot and Stein [1998]; Rampini, Sufi, and Viswanathan [2014]). In a multi-period context, this effective risk aversion will affect the valuation of future cash flows. Therefore, one primary take-away of the above is the inconsistency of the canonical model resulting in the familiar marginal cost (13): The motivation for holding and allocating capital is company risk aversion—which in turn should also be reflected in the valuation of cash flows. The key issue with the result in the presence of company risk aversion (14), however, is the exogenous specification of the company’s utility function \( U \). In our setting, risk aversion emerges through the mechanics contemplated in the theoretical risk management literature, even though the assessment of profits is ex-ante risk-neutral in Equation (6). In particular, the form of the weighting function, and, thus, the company’s effective preferences, emerge endogenously in our setting.

Therefore, RAROC-type calculations are still possible when accounting for the company’s adjusted valuation. We can rewrite the marginal cost equation (11) as:

\[
\text{RAROC}_i = \frac{\text{MR}_i - \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} w(I) 1_{\{I \leq D\}} \right]}{\frac{\partial}{\partial q^{(i)}} \text{VaR}_\psi(I)} = \mathbb{E} \left[ w(I) 1_{\{I>D\}} \right],
\]

(15)

so that the marginal returns on risk capital still equate the hurdle rate \( \mathbb{E}[w(I) 1_{\{I>D\}}] \) at the optimum. However, as in the expressions for marginal cost, the calculated return here is adjusted accounting for the company’s effective risk aversion. That is, not only the capital in the denominator is risk-adjusted, but there is also a risk-adjustment to the numerator as well as to the hurdle rate. It is interesting to note that the risk-adjustments in the numerator here originate from company risk aversion whereas the adjustment in the denominator originates from consumer risk aversion—which in our case is captured by the risk functional in the premium function.

Implementation of this RAROC ratio thus requires a specification of these adjustments. For

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10 This documents the motivation for—and the futility of—coming up with multi-period risk adjustments in single-period marginal cost equations in practical solvency frameworks (Möhr [2011]).
evaluating company risk from the consumer’s perspective, the conventional approach is to rely on a risk measure—VaR in our case. Risk adjustment from the company’s perspective, as captured by the weighting function \( w \), requires a solution of the company’s optimization problem, which entails optimal capitalization and portfolio decisions. In other words, a “short-cut” approach to pricing and performance measurements via the return ratio will only be exact when all inputs are available—which in turn would make the return ratio redundant. Whether the RAROC ratio is viable for practical purposes depends on the empirical question of whether feasible approximations—e.g., that ignore the risk adjustments as in (13) or that use an approximation in the form of a company utility function as in (14)—are sufficiently accurate to reflect the company’s risk situation.

3 Implementation in the Context of a Catastrophe Insurer

In this section, we calibrate and numerically solve the model introduced in the previous section using data from a CAT reinsurer, where we are focusing on two questions: (i) What is the shape of the company’s effective preference function; and (ii) how do the results generated from our model compare to RAROC from conventional methods.

We describe the data and our aggregation to four business lines, calibration based on industry data, and implementation in Section 3.1. Section 3.2 presents our results. Default is a very low probability event. The company holds capital to shield from default, and under optimal capital level makes use of the emergency financing option in about 0.5% of all scenarios. We find that the value of the firm as a function of capital is concave with an optimal capitalization point that trades off profitability and (re-)capitalization costs. The optimal raising decision essentially pushes capital to the optimal point, although it is rigid in the area around the optimum due to a difference in the cost of shedding (nil) and raising (positive) capital. The optimal risk portfolio increases convexly up to a saturation point, after which the portfolio is kept constant and excess capital is shed (down to the saturation point).

In Section 3.3 we calculate allocations of capital to different risks and different cost components, finding significant differences between correctly calculated dynamic RAROCs and their conventional static counterparts. The hurdle rate is lower than the cost of external finance due to the possibility of optimally combining different capitalization options—and it may even be less than the cost of internal capital due to benefits with regards to the firm’s continuation value. On the other hand, the amount of “capital” to be allocated, which includes franchise value in our approach, is significantly greater than the usual capital metrics used in conventional approaches. Thus, the level of dynamic RAROCs is typically much lower than the static figures, though the hurdle rate is also lower. We also find that capital costs are generally the most important cost component after actuarial costs, but that risk adjustments to actuarial costs can be a considerable portion of
total costs for firms with low capital levels. To illustrate the nature of the risk adjustments, we derive the company’s effective utility function. The company’s relative risk aversion exhibits an inverse U-shape, where we find a maximal relative risk aversion of roughly 20%. Accounting for the cost of emergency raising and company effective risk aversion using a constant relative risk aversion (CRRA) assumption of 12% (the weighted average) delivers RAROCs that align closely with the optimal values, suggesting that relatively simple modifications to conventional RAROC approaches can yield practical improvements.

Additional details on calibration are given in Appendix A.3. Details on implementation and results illustrating the convergence of the numerical algorithm are provided in Appendix A.4. Appendix B collects additional results.

3.1 Data, Calibration, and Implementation

We are given 50,000 joint loss realizations and premiums for 24 distinct reinsurance lines differing by peril and geographical region. The data have been scaled by the data supplier. Figure 1 provides a histogram of the aggregate loss distribution, and Table 1 lists the lines and provides some descriptive statistics for each line.

We aggregate the data to four lines and in what follows focus on the problem of optimally allocating to these aggregated lines. This has the advantage of keeping the numerical analysis
<table>
<thead>
<tr>
<th>Line</th>
<th>Premiums</th>
<th>Expected Loss</th>
<th>Standard Deviation</th>
<th>Agg</th>
</tr>
</thead>
<tbody>
<tr>
<td>N American EQ East</td>
<td>6,824,790.67</td>
<td>4,175,221.76</td>
<td>26,321,685.65</td>
<td>1</td>
</tr>
<tr>
<td>N American EQ West</td>
<td>31,222,440.54</td>
<td>13,927,357.33</td>
<td>47,198,747.52</td>
<td>1</td>
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<tr>
<td>S American EQ</td>
<td>471,810.50</td>
<td>215,642.22</td>
<td>915,540.16</td>
<td>1</td>
</tr>
<tr>
<td>Australia EQ</td>
<td>1,861,157.54</td>
<td>1,712,765.11</td>
<td>13,637,692.79</td>
<td>1</td>
</tr>
<tr>
<td>Europe EQ</td>
<td>2,198,888.30</td>
<td>1,729,224.02</td>
<td>5,947,164.14</td>
<td>1</td>
</tr>
<tr>
<td>Israel EQ</td>
<td>642,476.65</td>
<td>270,557.81</td>
<td>3,234,795.57</td>
<td>1</td>
</tr>
<tr>
<td>NZ EQ</td>
<td>2,901,010.54</td>
<td>1,111,430.78</td>
<td>9,860,005.28</td>
<td>1</td>
</tr>
<tr>
<td>Turkey EQ</td>
<td>214,089.04</td>
<td>203,495.77</td>
<td>1,505,019.84</td>
<td>1</td>
</tr>
<tr>
<td>N Amer. Severe Storm</td>
<td>16,988,195.98</td>
<td>13,879,861.84</td>
<td>15,742,997.51</td>
<td>2</td>
</tr>
<tr>
<td>US Hurricane</td>
<td>186,124,742.31</td>
<td>94,652,100.36</td>
<td>131,791,737.41</td>
<td>2</td>
</tr>
<tr>
<td>US Winterstorm</td>
<td>2,144,034.55</td>
<td>1,967,700.56</td>
<td>2,611,669.54</td>
<td>2</td>
</tr>
<tr>
<td>Australia Storm</td>
<td>124,632.81</td>
<td>88,108.80</td>
<td>622,194.10</td>
<td>2</td>
</tr>
<tr>
<td>Europe Flood</td>
<td>536,507.77</td>
<td>598,660.08</td>
<td>2,092,739.85</td>
<td>2</td>
</tr>
<tr>
<td>ExTropical Cyclone</td>
<td>37,033,667.38</td>
<td>23,602,490.43</td>
<td>65,121,405.35</td>
<td>2</td>
</tr>
<tr>
<td>UK Flood</td>
<td>377,922.95</td>
<td>252,833.64</td>
<td>2,221,965.76</td>
<td>2</td>
</tr>
<tr>
<td>US Brushfire</td>
<td>12,526,132.95</td>
<td>8,772,497.86</td>
<td>24,016,196.20</td>
<td>3</td>
</tr>
<tr>
<td>Australian Terror</td>
<td>2,945,767.58</td>
<td>1,729,874.98</td>
<td>11,829,262.37</td>
<td>4</td>
</tr>
<tr>
<td>CBNR Only</td>
<td>1,995,606.55</td>
<td>891,617.77</td>
<td>2,453,327.70</td>
<td>4</td>
</tr>
<tr>
<td>Cert. Terrorism xCBNR</td>
<td>3,961,059.67</td>
<td>2,099,602.62</td>
<td>2,975,452.18</td>
<td>4</td>
</tr>
<tr>
<td>Domestic Macro TR</td>
<td>648,938.81</td>
<td>374,808.73</td>
<td>1,316,650.55</td>
<td>4</td>
</tr>
<tr>
<td>Europe Terror</td>
<td>4,512,221.99</td>
<td>2,431,694.65</td>
<td>8,859,402.41</td>
<td>4</td>
</tr>
<tr>
<td>Non Certified Terror</td>
<td>2,669,239.84</td>
<td>624,652.88</td>
<td>1,138,937.44</td>
<td>4</td>
</tr>
<tr>
<td>Casualty</td>
<td>5,745,278.75</td>
<td>2,622,161.64</td>
<td>1,651,774.25</td>
<td>4</td>
</tr>
<tr>
<td>N American Crop</td>
<td>21,467,194.16</td>
<td>9,885,636.27</td>
<td>18,869,901.33</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1: Descriptive statistics for the loss profiles for each of the 24 business lines written by our catastrophe reinsurer. The data are scaled by our data supplier.
tractable and facilitates the presentation of results. Table 1 illustrates the aggregation (column Agg), and Figure 2 shows histograms for each of these four lines.

The “Earthquake” (Agg 1) distribution is concentrated at low loss levels with few realizations exceeding 50,000,000 (the 99% VaR lightly exceeds 300,000,000). However, the distribution depicts fat tails with a maximum loss realization of close to one billion. The (aggregated) premium for this line is 46,336,664 with an expected loss of 23,345,695. “Storm & Flood” (Agg 2) is by far the largest line, both in terms of premiums (243,329,704) and expected losses (135,041,756). The distribution is concentrated around loss realizations between 25 and 500 million, although the maximum loss in our 50,000 realizations is almost four times that size. The 99% VaR is approximately 700 million. In comparison, the “Fire & Crop” (Agg 3) and “Terror & Casualty” (Agg 4) lines are smaller with aggregated premiums (expected loss) of about 34 (19) million and 22.5 (11) million, respectively. The maximal realizations are around 500 million for “Fire & Crop” (99% VaR = 163,922,557) and around 190 million for “Terror & Casualty” (99% VaR = 103,308,358).

The model as developed in Section 2 requires calibration in several areas. It is necessary to specify costs of raising and holding capital. It is also necessary to specify how insurance premiums are affected by changes in risk. As is detailed in Appendix A.3, we rely on relevant literature for the calibration of capital costs, where we use specific results for insurance markets where available (Cox and Rudd, 1991; Cummins and Phillips, 2005) and more general estimates otherwise (Hennessy and Whited, 2007). For connecting risk and premiums, we rely on company ratings in conjunction with agencies’ validation studies in order to obtain default rates for U.S. reinsurance companies. We then estimate the parameters in our premium specification (5) using financial statement data between the years 2002 and 2010 as available from the National Association of Insurance Commissioners (NAIC, see Table 9 in Appendix A.3).

Based on this calibration exercise, we use various sets of parameters. We present results for three sets that are described in Table 2. We vary the cost of holding capital \( \tau \) from 3% to 5%; the cost of raising capital in normal circumstances is represented by a quadratic cost function with the linear coefficient \( c_1^{(1)} \) fixed at 7.5%; the cost of raising capital in distressed circumstances, \( \xi \), varies from 20% to 75%; the interest rate \( r \) varies from 3% to 6%; and for the parameters \( \alpha, \beta, \) and \( \gamma \), we use the regression results from Table 9 for our “base case,” with the alpha intercept being adjusted for the average of the unreported year dummy coefficients. In addition, we also use an alternative, more generous specification based on an analysis that omits loss adjustment expenses for parameter sets 2 and 3.

Using the loss distributions described in Section 3.1, we solve the optimization problem by value iteration relying on the corresponding Bellman equation (9) on a discretized grid for the capital level \( a \). That is, we commence with an arbitrary value function (constant at zero in our
case), and then iteratively solve the one-period optimization problem by using the optimized value function from the previous step on the right hand side. Standard results on dynamic programming guarantee the convergence of this procedure (Bertsekas, 1995). More details on the solution algorithm and its convergence are presented in Appendix A.4.

3.2 Results

The results vary considerably across the parameterizations. While the value function in the base case ranges from approximately 1.8 billion to 2 billion for the considered capital levels, the range for the “profitable company” is in between 21.7 billion to 22.4 billion, and even around 56 to 57 billion for our “empty company.” The basic shape of the solution is similar across the first two cases, whereas the “empty company case” yields a qualitatively different form (hence the name).
### Base Case Solution

Various aspects of the “base case” solution are depicted in Figures 3, 4, and 5. Table 3 presents detailed results at three key capital levels.

![Image of value function and its derivative](image)

(a) Value Function $V(a)$

(b) Derivative $V'(a)$

Figure 3: Value function $V$ and its derivative $V'$ for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\times10^{-10}$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48\times10^{-10}$ (base case).

Figure 3 displays the value function and its derivative. We observe that the value function is “hump-shaped” and concave—i.e., the derivative $V'$ is decreasing in capital. For high capital levels, the derivative approaches a constant level of $-\tau = -3\%$, and the value function is essentially affine.

The optimal level of capitalization here is approximately 1 billion. If the company has significantly less than 1 billion in capital, it raises capital as can be seen from Figure 4, where the optimal
raising decision for the company is displayed. However, the high and convex cost of raising external financing prevents the company from moving immediately to the optimal level. The adjustment can take time: Since internally generated funds are cheaper than funds raised from investors, the optimal policy trades off the advantages associated with higher levels of capitalization against the costs of getting there. As pointed out by Brunnermeier, Eisenbach, and Sannikov (2013), persistence of a temporary adverse shock is a common feature of models with financing frictions. As capitalization increases, there is a rigid region around the optimal level where the company neither raises nor sheds capital. In this region, additional capital may bring a benefit, but it is below the marginal cost associated with raising an additional dollar, which is approximately $c^{(1)}_1 = 7.5\%$. The benefit of capital may also be less than its carrying cost of $\tau = 3\%$, but since this cost is sunk in the context of the model, capital may be retained in excess of its optimal level. For extremely high levels of capital, however, the firm optimally sheds capital through dividends to immediately return to a maximal level at which point the marginal benefit of holding an additional unit of capital (aside from the sunk carrying cost) is zero. The transition is immediate, as excess capital incurs an unnecessary carrying cost and shedding capital is costless in the model. This is also the reason that the slope of the value function approaches $-\tau$ in this region.

Figure 5 shows how the optimal portfolio varies with different levels of capitalization. As capital is expanded, more risk can be supported, and the portfolio exposures grow in each of the lines until capitalization reaches its maximal level. After this point, the optimal portfolio remains constant: Even though larger amounts of risk could in principle be supported by larger amounts of capital, it is, as noted above, preferable to immediately shed any capital beyond a certain point.
and, concurrently, choose the value maximizing portfolio. Note that the firm here has an optimal scale because of the \( \gamma \) parameter in the premium function. As the firm gets larger in scale, margins shrink because of \( \gamma \).

Table 3 reveals that firm rarely exercises its default option (measured by \( \mathbb{P}(I \geq D) \), which is 0.002\% even at low levels of capitalization). The firm does experience financial distress more often at low levels of capitalization. For example, the probability of facing claims that exceed immediate financial resources, given by \( \mathbb{P}(I > S) \), is 4.54\% when initial capital is zero but 0.45\% when capital is at the optimal level, and 0.13\% when capitalization is at its maximal point. In all of these cases, the firm usually resorts to emergency financing when claims exceed its cash, at a per unit cost of \( \xi = 50\% \), to remedy the deficit. Because of the high cost of emergency financing, however, it restrains its risk taking when undercapitalized and also raises capital before underwriting to reduce the probability of financial distress.

The bottom rows of the table show the various cost parameters at the optimized value. Here, the marginal cost of raising capital, \( c_1'(R_b) \), is significantly greater than 7.5\% for \( a = 0 \) due to the quadratic adjustment, whereas clearly the marginal cost is zero in the shedding region (\( a = 4bn \)). As indicated above, around the optimal capitalization level of 1 billion neither raising nor shedding is optimal—so that technically the marginal cost is undefined due to the non-differentiability.
Table 3: Results for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00E-10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48E-10$ (base case).
of the cost function \( c_1 \) at zero. To determine the correct “shadow cost” of raising capital, we use an indirect method: We use the aggregated marginal cost condition (11) from Proposition A.3 to back out the value of \( c_1'(0) \) that causes the left- and right-hand side to match up. The cost of emergency raising in this case is exactly the probability of using this option (as \( \xi = 50\% \)), which—as indicated—decreases in the capital level. Finally, the expected cost in terms of impact on the value function \( -\mathbb{E}[V' 1_{I<s}] \) is negative for low capital levels since the value function is increasing in this region, whereas it is positive and approaching \( \tau \) for high capital levels. Combining the different cost components, we obtain a “hurdle rate” \( \mathbb{E}[w(I) 1_{I>D}] \) that only varies slightly across the different levels of capitalization. In particular, it is noteworthy that the hurdle rate is considerably below the marginal cost of raising capital. The next subsection provides a more detailed discussion of the marginal cost of risk.

**Profitable Company**

The results for the profitable company are similar to the “base case” presented above, except that the company is now much more valuable—despite the increases in the carrying cost of capital and in the cost of emergency financing—because of the more attractive premium function. The corresponding results are collected in Appendix B. More precisely, Figure 11 displays the value function and its derivative, Figure 12 displays the optimal raising decision, and Figure 13 displays the optimal exposure to the different lines as a function of capital.

Again, there is an interior optimum for capitalization, and the company optimally adjusts toward that point when undercapitalized. If overcapitalized, it optimally sheds to a point where the net marginal benefit associated with holding a dollar of capital (aside from the current period carrying cost which is a sunk cost) is zero. There is thus a rigid range where the company neither raises nor sheds capital, and the risk portfolio gradually expands with capitalization until it reaches the point where the firm is optimally shedding additional capital on a dollar-for-dollar basis.

As before, Table 4 presents detailed results at three key capital levels. Although parameters have changed, the company again rarely exercises the option to default, which still has a probability of occurrence of 0.002% even at low levels of capitalization. In most circumstances, the firm chooses to raise emergency financing when claims exceed cash resources, which happens as much as 3.65% of the time (at zero capitalization).

In contrast to the base case, the “hurdle rate” \( \mathbb{E}[w(I) 1_{I>D}] \) now is substantially larger. To some extent, this originates from the different cost parameters. In particular, the cost of raising emergency capital now is \( \xi = 75\% \) and the carrying cost \( \tau = 5\% \). However, in addition to

\[11\]In the differentiable regions \( (a = 0, 4b, \text{and other values}) \), the aggregated marginal cost condition further validates our results—despite discretization and approximation errors, the deviation between the left- and right-hand side is maximally about 0.025% of the left-hand side.
<table>
<thead>
<tr>
<th></th>
<th>zero capital</th>
<th>optimal capital</th>
<th>high capital</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>3,000,000,000</td>
<td>12,000,000,000</td>
</tr>
<tr>
<td>(V(a))</td>
<td>22,164,966,957</td>
<td>22,404,142,801</td>
<td>22,018,805,587</td>
</tr>
<tr>
<td>(R(a))</td>
<td>1,106,927,845</td>
<td>0</td>
<td>-6,102,498,331</td>
</tr>
<tr>
<td>(q_1(a))</td>
<td>4.81</td>
<td>6.14</td>
<td>7.82</td>
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<tr>
<td>(q_2(a))</td>
<td>4.42</td>
<td>5.64</td>
<td>7.18</td>
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<tr>
<td>(q_3(a))</td>
<td>9.83</td>
<td>12.56</td>
<td>15.98</td>
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<tr>
<td>(q_4(a))</td>
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<td>50.69</td>
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<tr>
<td>(S)</td>
<td>3,659,208,135</td>
<td>6,215,949,417</td>
<td>9,412,766,805</td>
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<tr>
<td>(D)</td>
<td>9,200,449,874</td>
<td>11,757,191,157</td>
<td>14,954,008,545</td>
</tr>
<tr>
<td>(\mathbb{E}[I])</td>
<td>1,227,901,222</td>
<td>1,569,126,466</td>
<td>1,995,776,907</td>
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<tr>
<td>(\sum p^{(i)}/\mathbb{E}[i])</td>
<td>2.15</td>
<td>2.03</td>
<td>1.90</td>
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<td>10.70%</td>
<td>0.07%</td>
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<td>0.34%</td>
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<tr>
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<td>0.002%</td>
<td>0.002%</td>
</tr>
<tr>
<td>(c'_{i}(R^a))</td>
<td>18.57%</td>
<td>5.97%</td>
<td>0.00%</td>
</tr>
<tr>
<td>(\frac{\xi}{1-\xi}\mathbb{P}(S &lt; I &lt; D))</td>
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<td>2.72%</td>
<td>1.00%</td>
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<tr>
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<td>-4.58%</td>
</tr>
<tr>
<td>(\mathbb{E}[u(I)\mathbb{I}_{I&gt;D}])</td>
<td>7.28%</td>
<td>6.22%</td>
<td>3.58%</td>
</tr>
</tbody>
</table>

Table 4: Results for a company with carrying cost \(\tau = 5\%\), raising costs \(c^{(1)} = 7.5\%\), \(c^{(2)} = 5.00E-11\), and \(\xi = 75\%\), interest rate \(r = 6\%\), and premium parameters \(\alpha = 0.9730\), \(\beta = 550.20\), and \(\gamma = 1.61E-10\) (profitable company).
higher costs, another aspect is that given the more profitable premium function, it now is optimal to write more business requiring a higher level of capital—which in turn leads to higher capital costs. Essentially, the marginal pricing condition [11] requires marginal cost to equal marginal return/profit—and the point where the two sides align now is at a higher level.

Empty Company

Figure 6 presents the value function and the optimal exposures to the different business lines for the “empty company.” We call this case the “empty company” because it is optimal to run the company without any capital. This can be seen from Figure 6(a), which shows that the total continuation value of the company is decreasing, so that the optimal policy is to shed any and all accumulated capital through dividends. The optimal portfolio is thus, as can be seen in Figure 6(b), always the same—corresponding to the portfolio chosen when \( a = 0 \). Again, there is an optimal scale in this case, as greater size is associated with a compression in margins.

![Value function and optimal portfolio weights](image-url)

Figure 6: Value function \( V(a) \) and optimal portfolio weights \( q_1, q_2, q_3, \) and \( q_4 \) for a company with carrying cost \( \tau = 5\% \), raising costs \( c^{(1)} = 7.5\% \), \( c^{(2)} = 1.00E-10 \), and \( \xi = 20\% \), interest rate \( r = 3\% \), and premium parameters \( \alpha = 0.9730 \), \( \beta = 550.20 \), and \( \gamma = 1.61E-10 \) (empty company).

However, even though the company is always empty, it never defaults. This extreme result is produced by two key drivers—the premium function and the cost of emergency financing. As with the “profitable company,” the premium function is extremely profitable in expectation. Because of these high margins, staying in business is extremely valuable. Usually, the premiums collected are sufficient to cover losses. When they are not, which happens about 12% of the time, the company resorts to emergency financing. This happens because, in contrast to the “profitable company,” emergency financing is relatively cheap at 20% (versus 75% in the “profitable company” case). Thus, it makes sense for the company to forego the certain cost of holding capital—the primary
benefit of which is to lessen the probability of having to resort to emergency financing—and instead just endure the emergency cost whenever it has to be incurred. In numbers, the cost of holding capital at \( a = 0 \) is \( \tau \times P(I \leq S) = 4.38\% \), whereas the cost of raising emergency funds is \( \frac{\xi}{1-\xi}P(I > S) = 3.08\% \).

3.3 The Marginal Cost of Risk and Capital Allocation

Typical capital allocation methods consider allocating assets \((S)\) or book value capital \((a)\). In contrast, as is detailed in Section 2.2 our model prescribes a broader notion of capital that considers all financial resources \((D)\). However, even if we identify the correct quantity to allocate, Equation (11) shows that then marginal cost of risk goes beyond that obtained from a simple allocation of \( D \) in two respects. First, calculating the cost of “capital” when allocating \( D \) is not straightforward: The theoretical analysis indicates that the key quantity is the risk-adjusted default probability \( \mathbb{E}[w(I)I_{\{I > D\}}] \) that accounts for the value of capital in default states. Second, the valuation of the company in different (loss) states reflected by the weighting function \( w(\cdot) \) will affect the determination of the “return” in the numerator of a RAROC ratio.

Base Case

Figure 7 plots the weighting function for the three capital levels considered in Table 3. According to the definition of \( w \) (Eq. (12)), the plots for each capital level exhibit two discontinuities at \( S \) and \( D \). For realizations less than \( S \), the weighting function equals:

\[
w(I) = \left(1 - c'_1\right) \times \left(1 + V'(S - I)\right).
\]

The latter term \((II)\) measures the marginal benefit of an additional dollar of loss-state-contingent income accounting for its impact on firm value, so that it can be interpreted as the company’s “marginal effective utility.” The former term \((I)\) reflects the firm’s marginal cost capital, since premiums charged by the company and capital are substitutes, so that it can be interpreted as the company’s “internal discount factor.” The weight \( w \) then is the product. In particular, for \( a = 0 \), marginal effective utility is high \((> 1)\) since additional capital carries a substantial benefit, but simultaneously the cost of capital is high so that the discounting will be substantial—overall yielding a weight of slightly less than one. In contrast, for high capital levels, the internal discount factor is one (since the company is shedding capital); the marginal effective utility, on the other hand, is less than one for low loss realizations due to (sunk) internal capital cost \( \tau \) but then increases above one in very high loss states since here the marginal effective utility exceeds one due to the positive impact of an additional dollar on firm value.
For realizations in between $S$ and $D$, the weight equals the adjusted cost of emergency raising:

$$w(I) = (1 - c'_1) \times 1 / (1 - \xi).$$

The latter term $1 / (1 - \xi)$ is the same for all capital levels and now provides the direct marginal benefit of state-contingent income due to avoiding the cost of emergency raising, so that it again measures marginal effective utility to the company. The former term again reflects the firm’s marginal cost capital, so that the penalty for emergency raising is lower for low capital levels because it avoids raising more external capital—which is particularly costly here.

While the weighting in high capital states is always appears to be larger, note that this is misleading since of course the probabilities of falling in the different ranges vary between the capital levels. For instance, as is clear from Table 3, the probability for falling in the emergency raising range $[S, D]$, where the weighting significantly exceeds one, is 4.5%, 0.45%, and 0.13% for $a = 0$, $a$ optimal, and $a$ large, respectively. Importantly, since the marginal benefit of an additional dollar raised is a dollar at the optimum, all the value functions will integrate to one.

To obtain a sense of the relevance of the different cost components, and particularly the risk adjustment due to the weighting function, Table 5 shows the decomposition of the aggregated marginal cost $\sum_{i=1}^N q^{(i)} \text{MR}_i$, where $\text{MR}_i$ is given by Equation (11), into three components: (i) the actuarial value of solvent payments ($\mathbb{E}[I I_{\{I \leq D\}}]$), (ii) the value adjustment due to the weighting function ($\mathbb{E}[I (w(I) - 1) I_{\{I \leq D\}}]$), and (iii) capital costs ($D \times [\mathbb{E}[w(I) I_{\{I > D\}}]$).
Table 5: Total marginal cost allocation for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00E-10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48E-10$ (base case).

In current practice, the second component is typically ignored, so that the optimal solution aligns marginal excess premiums (over actuarial values) with marginal capital costs for each line (see Eq. 13). This omission is relatively insignificant in well-capitalized states in the base case ($a = 1bn$ or $4bn$). Indeed, the risk adjustments, which amount to less than one percent of total cost, are dwarfed by capital costs, which amount to between $16\%$ and $19\%$ of total cost.

This can also be seen from corresponding RAROC ratios, which we present in Table 6. The first rows for all the capitalization levels show the correct dynamic RAROCs according to Equation 15, where the denominators are determined as VaR allocations of the default value $D$ and the numerators include the risk adjustment due to the weighting function. Due to the optimality criterion, the RAROCs for the different lines coincide and equal the hurdle rate $\mathbb{E}[w(I) 1_{I>D}] = 2.9\%$, $3.18\%$, and $2.54\%$ for the three capitalization levels (cf. Table 3).

The second rows for the three levels present the RAROC ignoring the risk adjustment in the numerator, but still allocating the correct quantity $D$—or, equivalently, using the correct default threshold in the VaR. At the optimal level ($a = 1bn$) and the high capital level ($a = 4bn$), omitting the risk adjustment in the numerator is not critical: The RAROCs across the different lines are still similar and close to the correct hurdle rate. These observations vindicate conventional capital allocation approaches that ignore the risk adjustments, with the caveat that it is important to allocate the correct quantity. Indeed, the levels differ significantly when following the more conventional practice of allocating assets $S$ or accounting capital $a$ (third and fourth rows for the three capital levels in Table 6).

The situation changes for the low capital level $a = 0$. Here the aggregate value of the value adjustments to the numerator amounts to more than $5\%$ of total cost, whereas the capital cost amounts to roughly $17\%$. The value adjustment now represents a significant portion of costs after actuarial
Table 6: RAROC calculations for a company with carrying cost \( \tau = 3\% \), raising costs \( c^{(1)} = 7.5\% \), 
\( c^{(2)} = 1.00E-10 \), and \( \xi = 50\% \), interest rate \( r = 3\% \), and premium parameters \( \alpha = 0.3156 \), 
\( \beta = 392.96 \), and \( \gamma = 1.48E-10 \) (base case).

value (roughly 30\%). Consequently, ignoring the value adjustment in the RAROC becomes mate-
rial, as can be seen in Table 6 for \( a = 0 \). In this case, the RAROCs differ by up to 60 basis points, 
so constructing the line portfolio on this basis would yield inefficient outcomes. For example, the 
RAROCs suggest boosting line 4 and retracting line 1 (RAROCs of 4\% vs. 3.4\%).

Omitting the value adjustments would not affect the relative order of RAROCs if the allocation 
of the total value adjustment to the different business lines were analogous to the allocation of 
capital. The fact that we observe significant differences in the relative order of the RAROCs implies 
that the two allocations deviate. The reason is that the allocations are driven by different 
properties of the risk distribution. More precisely, while capital allocations are tied to default (and 
therefore the loss distribution’s tail properties are relevant), risk weighting for value adjustments 
is influenced more heavily by the central part of the distribution. For example, we note that high 
realizations in business line 1 drive default scenarios, whereas business line 4 frequently shows 
high realizations in solvent scenarios. Assuming that the valuation adjustments follow the same 
pattern as capital allocation will therefore lead to material errors.

As detailed above, the origin of the risk adjustment in the numerator is company effective risk 
aversion [Froot and Stein 1998, Rampini, Sufi, and Viswanathan 2014]. As discussed in Section 
2.2 we obtain a similar expression (14) for the marginal cost of risk with a risk adjustment when 
endowing the company with an (exogenous) utility function in a one-period model. To analyze
the effective preferences of the company, we derive the endogenous utility function $U$ that delivers the correct risk adjustment in our model. In other words, we back out the $U$ that implements the “correct” marginal cost for our multi-period model in the context of a basic one-period model by equating the corresponding marginal cost equations (11) and (14). Figure 8(a) plots the resulting relative risk aversion $RRA(x) = -xU''(x)/U'(x)$ for our CAT reinsurer as a function of residual capital $D-I$ in the base case for a company with $a = 4\text{bn}$.

Risk aversion is zero (and, thus, the effective utility function is linear) in two ranges: (i) For $D-I < D-I$ (so that $I > S$), and (ii) for very large $D-I$ (so that $I$ is small). The first region (i) is the emergency raising region, where we imposed a linear cost of emergency raising—leading to a linear effective utility function. Thus, this observation has to be interpreted with care, since it relies on the model specification (and it would change if we imposed a convex cost of emergency raising). Furthermore, the function is non-differentiable at the breaking point $I = D$, so that risk aversion is not defined here. The second region (ii) is where the company is over-capitalized and sheds capital, though incurring internal capital costs. The slope in the utility function is $(1 - \tau)$ in this region.

In between these regions, the effective utility function exhibits curvature. The risk aversion is maximally 0.17 right around loss levels that will result in an optimal capitalization level of roughly 1bn in the next period. This is the region where the company operates most efficiently, so that deviations in either direction are costly and the company is averse to risk. Note that this level is small compared to relative risk aversion levels typically found for individuals. For smaller loss realizations (greater levels of $D-I$), risk aversion decreases as $V(a)$ becomes more linear. For

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---

---

12 The shape of the relative risk aversion is independent of the initial capital level, but the relevant range of outcomes $D-I$ is different (since $D$ differs).
greater loss realizations (lower levels of $D - I$), risk aversion is also smaller, so that effective risk aversion exhibits an inverse-U-shape, reflecting the fact that the benefits of a marginal positive outcome offset a marginal negative outcome since it will bring the company closer to an optimal capital level. This observation is related to the ideas by Rampini, Sufi, and Viswanathan (2014) that more constrained companies engage less in risk management in a multi-period setting, although the mechanism in their paper is different.

The relevance of effective risk aversion, or rather the weighting function associated with company effective risk aversion, is greater for low capital levels, since here the probability of a realization that puts the company in a low capital range is relatively high. As seen in Table 6, RAROC ratios differ notably when not accounting for the risk adjustment at low capitalization levels.

To assess whether a short-cut approach via a single period capital allocation model is feasible, we consider a reduced-form approach for RAROC where we incorporate the cost of emergency raising for loss realizations between $S$ and $D$, and we impose a weighting function implied by a constant relative risk-aversion (CRRA) utility function $u(x) = x^{1-\gamma} / (1 - \gamma)$. We calibrate the risk aversion level as a weighted average (according to the probability of loss realizations) of the endogenous risk aversion level from Figure 8(a), resulting in $\gamma = 0.12$ for $a = 0$. The results are provided in the last rows for the three capitalization levels of Table 6.

We find that this reduced-form approach works surprisingly well. The RAROCs for the different lines align almost perfectly for all capital levels, and they differ from the actual hurdle rate by only a few basis points. This suggests that these relatively minor modifications to RAROC can deliver efficient underwriting results even for low capital levels.

**Profitable and Empty Companies**

Overall, the results for the profitable company are qualitatively analogous, with a few important qualifications. First, while the shapes of the company’s effective utility function and of the corresponding effective relative risk aversion function appear similar when comparing the “base” and “profitable” companies, the profitable company exhibits greater levels of risk aversion. As seen in Figure 8(b), the effective relative risk aversion now peaks at around 0.35. There are two drivers for this difference. On the one hand, the premium function implies that selling insurance is more profitable, so that changes in exposure have more significant consequences. On the other hand, the capital cost parameters are larger in this case, rendering raising (or internal carrying) capital more costly.

As a consequence, the resulting weighting function $w$ differs more significantly across loss realization levels, and thus the proportion of the risk adjustment cost increases relative to the base case (see Figure 14 and Table 10 in Appendix B). In particular, the value adjustment component now amounts to 10.3%, 3.5%, and 1.5% of total costs for low, optimal, and high capital, respec-
Table 7: RAROC calculations, profitable company case.

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<tr>
<th>a = 0</th>
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<th>Line 2</th>
<th>Line 3</th>
<th>Line 4</th>
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<td>7.28%</td>
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</table>

(1 + ξ) across the loss domain. In particular, the only cost component beyond actuarial costs is the value adjustment, which is completely due to emergency raising, and amounts to 4.7% of total cost.
4 Conclusion

In this paper, we develop a multi-period model for an insurance company with multiple sources of financing and derive risk pricing results from the optimality conditions.

The model represents a step toward greater sophistication in firm valuation and risk pricing, but only a step. Other nuances—such as regulatory frictions and rating agency requirements—would merit consideration in a richer model. Moreover, calibration of any model would obviously have to be tailored to the unique circumstances of each firm. For example, different model specifications could favor different risk measures. Our setup was a favorable one for VaR rather than Expected Shortfall, a consequence rooted in our specification of the premium function. More realistic specifications would undoubtedly point the way to more complicated risk measures.

Nevertheless, the dynamic model, even before refinement, offers at least two important insights for current practice rooted in static concepts. First, capital must be defined broadly to include some notion of franchise or “continuation” value; this is a theoretical point that cannot be derived or quantified in a static model, yet it is of important practical significance for both solvency assessment and risk pricing. Second, the risk aversion of the firm is not fully captured through the allocation of capital, as is implicitly assumed in the typical RAROC approach; additional modifications to the valuation of the payoffs associated with an exposure are necessary.

References


Appendix

A Technical Appendix

A.1 Elaboration of Results for the Multi-period Model

The following lemma shows that the optimization can be recast as a maximization of the discounted present value of future dividends:

Lemma A.1. The objective function (Eq. 6) may be equivalently represented as:

\[
\max_{\{p_t^{(j)}, q_t^{(j)}, R_t^b, R_t^e\}} \left\{ \mathbb{E} \left[ \sum_{t \leq t^*} \{a_{t^*+1} \geq 0, a_{t^*+1} \geq 0, a_{t^*+1} < 0 \} e^{-rt} \left[ -e^{r}\sum_j p_t^{(j)} - e^{r}\sum_j I_t^{(j)} - (\tau a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e) \right] \right] - a_0 \right\}. \tag{16}
\]

Proof. With the budget constraint (2):

\[
e^{-rt} a_t - e^{-r(t-1)} a_{t-1} - e^{-rt}[e^{r} R_t^b + R_t^e]
\]

\[
= e^{-rt} \left[ e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} - (\tau a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e) \right].
\]

Hence, the sum in \((6)\) can be written as:

\[
\sum_{t=1}^{\infty} I_{\{a_1 \geq 0, ..., a_t \geq 0\}} e^{-rt} \left[ e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} - (\tau a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e) \right]
\]

\[
- \sum_{\{t < t^*; a_1 \geq 0, a_{t-1} \geq 0, a_{t+1} < 0\}} e^{-rt} \left[ (a_{t^*+1} + R_t^b)e^r + R_t^e \right]
\]

\[
- e^{-rt^*} \left[ (a_{t^*+1} + R_t^b)e^r + R_t^e \right]
\]

\[
= \sum_{t \leq t^*} e^{-rt} \left[ -e^{r} R_t^b - R_t^e \right] + e^{-r(t^*-1)} a_{t^*-1} - a_0 - e^{-r(t^*-1)} a_{t^*-1}
\]

\[
= \sum_{t \leq t^*} e^{-rt} \left[ -e^{r} R_t^b - R_t^e \right] - a_0,
\]

which completes the proof. \(\square\)

We now proceed to express the problem as a Bellman equation. Denote the optimal value function, i.e., the solution to \((6)\) or \((16)\), by \(V(a_0)\)\(^{13}\). Then, under mild conditions on the loss dis-
tributions, the value function is finite and—as the solution to a stationary infinite-horizon dynamic programming problem—satisfies the following Bellman equation:

**Proposition A.1 (Bellman Equation).** Assume $r, \tau > 0$. Then the value function $V(\cdot)$ satisfies the following Bellman equation:

$$V(a) = \max_{\{p^{(j)}\}, \{q^{(j)}\}, R^b, R^e} \left\{ \begin{array}{c}
\mathbb{E}\left[ I((a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)})e^r+R^e-c_2(R^e) \geq I) \times \\
\sum_j p^{(j)} - e^{-r} I - \tau a - c_1(R^b) - e^{-r} c_2(R^e) \\
+ e^{-r} V \left( [a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)})e^r+R^e-c_2(R^e) - I] \right) \\
- I((a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)})e^r+R^e-c_2(R^e) < I) \left( a + R^b + e^{-r} R^e \right) \right] \right\},
\right.$$  

subject to (3); (5); $\{p^{(j)}\}, \{q^{(j)}\}$, $R^b_t \in \mathbb{R}$; and $R^e_t \geq 0$ is $\sigma(L^{(j)}, j = 1, \ldots, N)$-measurable.

**Proof.** Notice that our per-period profit function in (6) is bounded from above, so the Bellman equation follows from classical infinite-horizon dynamic programming results (see e.g. Proposition 1.1 in Bertsekas [1995, Chap. 3]).

Note that since raising capital at the end of the period—which we interpret as raising capital in distressed states—is more costly than raising capital under “normal” conditions (Eq. (1)), it only makes sense to either raise exactly enough capital to save the company or not to raise any capital at all: Raising more will not be optimal since it is possible to raise at the beginning of the next period on better terms; raising less will not be optimal since the company will go bankrupt and the policyholders are the residual claimants. This yields:

**Proposition A.2.** The optimal $R^e \in \{0, R^e_* \}$, where $R^e_*$ solves:

$$\sum_j I^{(j)} - \left[ a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r = R^e_* - c_2(R^e_*).$$

More precisely:

- for $\left[ a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r \geq I$, we have $R^e = 0$;

- for $\left[ a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r < I$ and $V(0) \geq R^e_*$, we have $R^e = R^e_*$;

- and for $V(0) < R^e_*$, we have $R^e = 0$.

yields higher profits.
Proof. Let

\[ a' = \left[ a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r + R^e - c_2(R^e) - \sum_j I^{(j)}; \]

then, conditional on \( a' < 0 \), the objective function is decreasing in \( R^e \) so that zero is the optimal choice. Conditional on \( a' > 0 \), on the other hand, decreasing \( R^e \) by a (small) \( \varepsilon > 0 \) and increasing \( R^b \) in the beginning of the next period will be dominant (since \( c'_2 > c'_1 \)), so \( R^e > 0 \) cannot be optimal. Finally, if \( a' = 0 \) and \( R^e > 0 \), then \( R^e = R^e_* \).

Moreover,

\[ - \left( a + R^b \right) \leq \sum_j p^{(j)} - e^{-r} \sum_j I^{(j)} - \tau a - c_1(R^b) - e^{-r} c_2(R^e_*) \]

\[ + e^{-r} V \left( \left[ a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r + R^e_* - c_2(R^e_*) - \sum_j I^{(j)} \right) \]

\[ \iff V(0) \geq - \left( \sum_j p^{(j)} + (1 - \tau)a + R^b - c_1(R^b) \right) e^r + \sum_j I^{(j)} + c_2(R^e_*) \]

\[ \iff V(0) \geq R^e_*, \]

which proves the last assertion. \( \square \)

The latter assertion states that it is optimal to only save the company if the (stochastic) amount of capital to be raised at the end of the period is smaller than the value of the company, i.e., if the investment has a positive net present value. For a linear specification of end-of-period costs, this leads to the following simplification of the optimization problem:

**Corollary A.1.** For linear costs \( c_2(x) = \xi x, \ x \geq 0 \):

\[ R^e_* = \frac{1}{1 - \xi} \left[ I - \left( a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right) e^r \right] = \frac{1}{1 - \xi} \left[ I - S \right], \]

where we define:

\[ S = \left[ a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r \]

and:

\[ D = \left[ a(1 - \tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r + (1 - \xi)V(0) \]
as the thresholds of \( I \) for saving—or, rather, being required to save—the company and for letting it default, respectively. Then the Bellman equation becomes:

\[
V(a) = \max_{\{p^{(i)}, q^{(i)}\}, R^b} \left\{ \mathbb{E} \left[ e^{-r} \left( I_{\{I \leq S\}} \times (S - I) + V(S - I) + I_{\{S < I \leq D\}} \frac{1}{1-\xi} [D - I] \right) \right] \right\} \\
- [a + R^b]
\]

subject to:

\[
p^{(i)}(i) = e^{-r} \mathbb{E} \left[ I^{(i)} \right] \times \exp \left\{ \alpha - \gamma \mathbb{E}[I] - \beta \phi(I, D) \right\}.
\]

We now analyze the optimality condition emerging from the Bellman equation with linear costs. Note that here we rely on the specification of the premium function with the general risk metric (4), whereas the expressions in the main text rely on the particular case of the default probability (5).

**Proposition A.3.** We have for the marginal revenue for risk \( i \):

\[
MR_i = \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}(i)} \right] \exp \left\{ \alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I] \right\} (1 - \gamma \mathbb{E}[I])
\]

\[
= \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}(i)} w(I) I_{\{I \leq D\}} \right] + \left[ \frac{\partial}{\partial q^{(i)}(i)} \phi(I, D) \times \frac{1}{-\phi_D(I, D)} \right] \times \mathbb{E} \left[ w(I) I_{\{I > D\}} \right],
\]

where \( \rho \) is the risk measure associated with the risk metric \( \phi \) that adds up:

\[
\sum_i q^{(i)} \left[ \frac{\partial \rho}{\partial q^{(i)}(I)} (I) \right] = \sum_i q^{(i)} \left[ \frac{\partial \phi(I, D)}{\partial q^{(i)}(I)} \right] = D,
\]

and the weighting function:

\[
w(I) = \begin{cases} 
(1 - c'_1(R^b)) \times (1 + V'(S - I)) & , I \leq S, \\
(1 - c'_1(R^b)) \times \frac{1}{1-\xi} & , S < I \leq D, \\
-\phi_D(I, D) e^{\epsilon} \sum_i p^{(i)} & , I > D,
\end{cases}
\]

satisfies \( \mathbb{E}[w(I)] = 1 \).
Proof. The first order conditions from the Bellman equation (17) are:

\[
\begin{align*}
\{q_i\} & - e^{-r} \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} (1 + V'(S - I)) I_{\{t \leq S\}} \right] - \frac{e^{-r}}{1 - \xi} \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} I_{\{S < t \leq D\}} \right] \\
& - \sum_{k \neq i} \lambda_k \left( \beta \frac{\partial \phi}{\partial q^{(i)}} + \gamma \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \right) e^{-r} \mathbb{E}[I^{(k)}] \exp \{\alpha - \gamma \mathbb{E}[I] - \beta \phi(I, D)\} \\
& + \lambda_i \left( e^{-r} \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] - \beta \frac{\partial \phi}{\partial q^{(i)}} \mathbb{E}[I^{(i)}] e^{-r} - \gamma \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \mathbb{E}[I^{(i)}] e^{-r} \right) \\
& \times \exp \{\alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I]\} = 0,
\end{align*}
\]

\[
\begin{align*}
\{p_i\} & \mathbb{E} \left[ (1 + V'(S - I)) I_{\{t \leq S\}} \right] + \frac{1}{1 - \xi} \mathbb{E} \left[ I_{\{S < t \leq D\}} \right] \\
& - \lambda_i + \sum_k \lambda_k \mathbb{E}[I^{(k)}] \beta \phi_D(I, D) \exp \{\alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I]\} = 0,
\end{align*}
\]

\[
\begin{align*}
[R^b] & \mathbb{E} \left[ (V'(S - I) (1 - c'_1(R^b)) - c'_1(R^b)) I_{\{t \leq S\}} \right] \\
& + \mathbb{E} \left[ \left( \frac{1}{1 - \xi} - \frac{c'_1(R^b)}{1 - \xi} - 1 \right) I_{\{S < t \leq D\}} \right] - \mathbb{P}(I > D) \\
& + \sum_k \lambda_k \mathbb{E}[I^{(k)}] \beta \phi_D(I, D) (1 - c'_1(R^b)) \exp \{\alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I]\} = 0.
\end{align*}
\]

Now due to the homogeneity property of \( \phi \), we obtain:

\[
0 = \frac{\partial}{\partial a} \phi(a I, a D) = D \phi_D(a I, a D) + \sum_{i=1}^n q^{(i)} \frac{\partial}{\partial q^{(i)}} \phi(a I, a D)
\]

\[
\Rightarrow D = \frac{\sum_i q^{(i)} \left( \frac{\partial}{\partial q^{(i)}} \phi(I, D) \right)}{-\phi_D(I, D)},
\]

i.e. the adding-up property. From \([p_i]\) and \([R^b]\):

\[
\lambda_i = - \sum_k \lambda_k \mathbb{E}[I^{(k)}] \beta \phi_D(I, D) \exp \{\alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I]\} + \frac{1}{1 - \xi} \mathbb{E} \left[ I_{\{S < t \leq D\}} \right] \\
+ \mathbb{E} \left[ (1 + V'(S - I)) I_{\{t \leq S\}} \right] \\
= - \mathbb{E} \left[ \left( \frac{\xi}{1 - \xi} - \frac{c'_1(R^b)}{1 - \xi} \right) I_{\{S < t \leq D\}} \right] - \frac{1}{1 - c'_1(R^b)} \mathbb{E} \left[ \left( \frac{\xi}{1 - \xi} - \frac{c'_1(R^b)}{1 - \xi} \right) I_{\{S < t \leq D\}} \right] + \mathbb{P}(I > D) \frac{1}{1 - c'_1(R^b)} + \mathbb{P}(I \leq D) \frac{1}{1 - c'_1(R^b)} \\
= \frac{\mathbb{P}(I \leq D)}{1 - c'_1(R^b)} + \frac{\mathbb{P}(I > D)}{1 - c'_1(R^b)} \\
= \frac{1}{1 - c'_1(R^b)}.
\]
Then with $[q_i]$, we then obtain:

$$
\begin{align*}
\mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \exp \{ \alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I] \} (1 - \gamma \mathbb{E}[I]) \\
= \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \left( 1 + V'(S - I) \right) \left( 1 - c'(R^b) \right) I_{\{I \leq S\}} + \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \left( 1 - \xi \right) \left( 1 - c'(R^b) \right) I_{\{S < I \leq D\}} \\
+ \beta \frac{\partial}{\partial q^{(i)}} \phi(I, D) \mathbb{E}[I] \exp \{ \alpha - \beta \phi(I, D) - \gamma \mathbb{E}[I] \}
\end{align*}
$$

Moreover, from $[R^b]$ we immediately obtain:

$$
\begin{align*}
\mathbb{E} \left[ w(I) I_{\{I \leq S\}} \right] + \mathbb{E} \left[ w(I) I_{\{S < I \leq D\}} \right] + \mathbb{E} \left[ w(I) I_{\{I \leq D\}} \right] = 1.
\end{align*}
$$

Note that the expressions in the main text follow since for $\phi(I, D) = \mathbb{P}(I > D)$:

$$
\frac{\partial}{\partial q^{(i)}} \phi(I, D) \times \frac{1}{-\phi_D(I, D)} = \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \bigg| I = D \right] = \partial / \partial q_i \text{VaR}_\psi(I).
$$

### A.2 A One-period Model for Capital Allocation

#### RAROC for a Risk-Neutral Insurer

A one period version of the profit maximization problem (6) takes the form:

$$
\max_{S, \{q^{(i)}\}} \left\{ \sum_j p^{(j)} - \mathbb{E} \left[ I I_{\{I \leq S\}} \right] - S \mathbb{P}(I > S) - c_1(S) \right\}
$$

with total indemnity payment $I = \sum_{j=1}^{N} I^{(j)} = \sum_{j=1}^{N} I^{(j)}(q^{(j)})$ and premium function:

$$
p^{(j)} = \mathbb{E} \left[ j^{(j)} \right] \exp \{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \},
$$

where to ease exposition we frame the problem as choosing assets $S$ rather than capital and asset costs $c_1(\cdot)$ rather than capital costs.
The first-order conditions are:

\[[q_i] \quad E \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \exp \{\alpha - \beta P(I > S) - \gamma E[I]\} \begin{cases} (1 - \gamma E[I]) \\ -\beta E[I] \exp \{\alpha - \beta P(I > S) - \gamma E[I]\} \frac{\partial P(I > S)}{\partial q^{(i)}} - E \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} 1_{I \leq S} \right] = 0, \\ \end{cases} \]

\[[S] \quad -\beta E[I] \exp \{\alpha - \beta P(I > S) - \gamma E[I]\} \frac{\partial P(I > S)}{\partial S} = -f_I(S) \]

where we denote by \( f_I \) the density function of \( I \). Now:

\[\frac{\partial P(I > S)}{\partial q^{(i)}} = E \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} I = S \right] f_I(S),\]

and the first part of the latter term is simply the partial derivative of the Value-at-Risk (VaR) with critical level \( \alpha = P(I > S) \) (Gourieroux, Laurent and Scaillet, 2000). Hence, by combining \([q_i]\) and \([S]\), we obtain:

\[\begin{align*}
\text{marginal revenue (MR}_i) & = E \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \exp \{\alpha - \beta P(I > S) - \gamma E[I]\} \begin{cases} (1 - \gamma E[I]) \\ -\beta E[I] \exp \{\alpha - \beta P(I > S) - \gamma E[I]\} \frac{\partial P(I > S)}{\partial q^{(i)}} + \frac{\partial \text{VaR}_I(I)}{\partial q^{(i)}} \times \left[ P(I > S) + c'_1(S) \right] \end{cases} \\
\text{marg. act. cost (MAC}_i) & = \text{cap. alloc.} \\
\text{marg. cap. cost (CoC)} & \end{align*}\]

Or, by rewriting, we obtain the familiar RAROC representation:

\[\text{RAROC}_i = \frac{\text{MR}_i - \text{MAC}_i}{\frac{\partial \text{VaR}_I(I)}{\partial q^{(i)}}} = \left[ P(I > S) + c'_1(S) \right] \text{CoC}.\]

**RAROC for a Risk-Averse Insurer**

Consider the same setting as above but now assume the insurer is risk-averse and evaluates profits according to a utility function \( U(\cdot) \):

\[\max_{S, \{q^{(j)}\}} \left\{ E \left[ U \left( \sum_j p^{(j)} 1_{I \leq S} - S 1_{I > S} - c_1(S) \right) \right] \right\}\]
with the same premium function (18). Then the first-order conditions become:

\[ q_i \quad \mathbb{E} \left[ u' \left( \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \exp \{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \} \right) (1 - \gamma \mathbb{E}[I]) \right] - \beta \mathbb{E}[I] \exp \{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \} \frac{\partial \mathbb{P}(I > S)}{\partial q^{(i)}} \right] - \mathbb{E} \left[ u' \frac{\partial I^{(i)}}{\partial q^{(i)}} 1_{(I \leq S)} \right] = 0, \]

\[ S \quad -\mathbb{E} \left[ u' \left( \beta \mathbb{E}[I] \exp \{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \} \frac{\partial \mathbb{P}(I > S)}{\partial S} - 1_{(I > S)} - c'_1(S) \right) \right] = 0, \]

so that Equation (19) changes to:

\[
\mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \exp \{ \alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I] \} (1 - \gamma \mathbb{E}[I]) \\
= \mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \frac{\partial \text{VaR}_o(I)}{\partial q^{(i)}} I_{(I \leq S)} + \frac{\partial \text{VaR}_o(I)}{\partial q^{(i)}} \left[ \mathbb{E} \left[ w(I) \right] I_{(I > S)} \right] + c'_1(S) \\
= \mathbb{E} \left[ w(I) \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] + \frac{\partial \text{VaR}_o(I)}{\partial q^{(i)}} \left[ \mathbb{E} \left[ w(I) \right] I_{(I > S)} \right] + c'_1(S) \\
\]

and

\[
\mathbb{E} \left[ \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] e^{\alpha - \beta \mathbb{P}(I > S) - \gamma \mathbb{E}[I]} (1 - \gamma \mathbb{E}[I]) - \mathbb{E} \left[ w(I) \frac{\partial I^{(i)}}{\partial q^{(i)}} \right] I_{(I \leq S)} \\
\frac{\partial \text{VaR}_o(I)}{\partial q^{(i)}} \\
\text{RAROC}_c \\
\]

\[
= \mathbb{E} \left[ w(I) I_{(I > S)} \right] + c'_1(S). 
\]

Therefore, the key differences to the marginal cost equation for a risk-neutral company (19) are (i) the indemnity payments in solvent states are not valued according to their actuarial cost but state-weights according to the company’s preferences enter the valuation; and (ii) similarly, the default probability is not assessed under its statistical probability \( \mathbb{P}(I > S) \) but the corresponding loss states of the worlds are weighted by \( w(I) \).

### A.3 Calibration Details for the Model in Section 3

#### Capital Cost Parameters

As a starting point for the costs of holding capital, [Cummins and Phillips (2005)](https://doi.org/10.1002/jae.745) estimate the cost of equity capital for insurance companies using data from the 1997-2000 period. They use several methods to derive a variety of estimates, including a single factor CAPM and Fama-French
three-factor cost of capital model (Fama and French, 1993). The estimates for property-casualty insurance fall in the neighborhood of 10% to 20%. Given that the risk-free interest rate used in the analysis was based on the 30-day T-Bill rate, which averaged about 5% over the sample period, the estimates suggest a risk premium for property-casualty insurance ranging from as little as 5% to as much as 15%. However, previous research has found unstable estimates of the cost of capital, suggesting that the risk premium may be considerably smaller (Cox and Rudd, 1991). Given the range of results, we use \( \tau \) ranging from 3% to 5% in the model.

Calibrating the cost of raising capital is more difficult, as we are not aware of studies specific to the property-casualty industry. Hennessy and Whited (2007), however, analyze the cost of external financing average across industries by using the entire sample of Compustat firms. They find marginal equity flotation costs ranging from 5% for large firms to 11% for small firms, and we base our quadratic specification on these figures, with the linear piece \( (c_1^{(1)}) \) being set at 7.5%.

### Connecting Risk and Premiums

Changes in risk are known to affect insurance companies. Epermanis and Harrington (2006) focus on the property-casualty insurance industry in particular, documenting significant declines in premium growth following rating downgrades. Sommer (1996) documents a significant connection between default risk and pricing in the property-casualty industry. The foregoing research suggests two possible ways to model the consequences of risk for a property-casualty insurer: Increases in risk could either produce involuntary drops in exposure volume or drops in price, or both. We incorporate both channels in our model setup within the premium function (Eq. (5)).

We use credit ratings to define risk. However, while credit ratings are widely accepted proxies for market assessments of a company’s risk level, using them requires us to map credit ratings to default risk levels, which is a feasible exercise given the validation studies provided by rating companies that document the historical connection between default risk and the various letter ratings.

To calculate the default rate, we use a multi-stage procedure. We start by collecting (i) Moody’s, S&P, and A.M. Best ratings from the 2008-2012 period for the sample of insurance companies; (ii) the joint distribution of Moody’s and S&P ratings for corporate debt as reported in Table 1 of Cantor, Packer, and Cole (1997); and (iii) one year default rates by rating as reported in Tables 34 and 35 of Moody’s Annual Default Study: Corporate Default and Recovery Rates, 1920-2012 and Tables 9 and 24 of S&P’s 2012 Annual Corporate Default Study and Rating Transitions. We then fit smoothed default rates for Moody’s by choosing default rates for the AA1, AA2, AA3,
A1, A2, and A3 categories (AAA, BAA1, and other historical default rates are held at their historic values\(^{14}\)) and perform a similar procedure for S&P ratings. We calculate an average one-year default rate for A++, A+, A, and A- A.M. Best ratings by calculating an average “Moody’s” default rate based on our sample distribution of Moody’s ratings for each A.M. Best rating, calculating an average “S&P” default rate in a similar manner, and then averaging the two. This yields one-year default rates for each A.M. Best rating in the 2008-2012 sample as shown in Table 8\(^{15}\).

<table>
<thead>
<tr>
<th>Rating</th>
<th>Default Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>A++</td>
<td>0.006%</td>
</tr>
<tr>
<td>A+</td>
<td>0.044%</td>
</tr>
<tr>
<td>A</td>
<td>0.072%</td>
</tr>
<tr>
<td>A-</td>
<td>0.095%</td>
</tr>
</tbody>
</table>

Table 8: Fitted One-Year Default Rates for A.M. Best Ratings

The question of how to connect risk with pricing is an empirical one, requiring an analysis of the historical relation between default risk inferred from credit ratings and insurance prices. Since the data used for our numerical analysis is drawn from a reinsurance company, we focus on empirical analysis of reinsurers, and specifically those identified in the Reinsurance Association of America’s annual review of underwriting and operating results for the years 2008-2012. These reviews yield 30 companies for the analysis, and we collected all available ratings for that set of 30 companies from Moody’s, S&P, and A.M. Best.

We identify the relation between price and default risk and volume by fitting the model:

\[
\log p_{it} = \alpha + \alpha_t - \beta d_{it} - \gamma E_{it} + e_{it},
\]

where \(p_{it}\) is calculated as the ratio of net premiums earned to the sum of loss and loss adjustment expenses incurred by company \(i\) in year \(t\), \(d_{it}\) is the default rate corresponding to the letter rating of company \(i\) in year \(t\) in percent, \(E_{it}\) is the expected loss of company \(i\) in year \(t\), and \(e_{it}\) is an error.

\(^{14}\)Fit is assessed by evaluating 8 measures: 1) the weighted average default rate in the Aa category (using the modifier distribution in Cantor, Packer, and Cole (1997) for the weights); 2) the weighted average default rate in the A category, and “fuzzy” default rates for Aa1, Aa2, Aa3, A, A1, A2, and A3 categories, where the fuzzy rate is calculated by applying the distribution of S&P ratings for each modified category to the default rates (for example, if S&P rated 20% of Moody’s Aa1 as AAA, 50% as AA+ and 30% as AA, we would calculate the “fuzzy” default rate for Aa1 as 20%×Aaa default rate + 50%×Aa1 default rate + 30%×Aa2 default rate). We calculate squared errors between fitted averages and averages using the actual empirical data, and select fitted values to minimize the straight sum of squared errors over the eight measures.

\(^{15}\)It is worth noting that these are somewhat lower than suggested by A.M. Best’s own review of one-year impairment rates, which indicated 0.06% for the A++/A+ category and 0.17% for the A/A- category (see Exhibit 2 of Best’s Impairment Rate and Rating Transition Study – 1977 to 2011). In the empirical analysis that followed, it was also necessary to assign a default rate for the B++ rating, which did not occur in the 2008-2012 sample but did surface when performing the analysis over a longer time period. We used 0.20% for this default rate, which is roughly consistent with the default rates for Baa or BBB ratings.
term. The expected loss is calculated by applying the average net loss and loss adjustment expense ratio over the sample period for each firm to that year’s net premium earned.

We use NAIC data for the period 2002 to 2010 for the sample of companies identified above for the analysis. The results of the regression are presented in Table 9.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept ($\alpha$)</td>
<td>0.6590</td>
<td>0.0614</td>
<td>10.73</td>
</tr>
<tr>
<td>Default rate ($\beta$)</td>
<td>3.9296</td>
<td>0.5090</td>
<td>-7.72</td>
</tr>
<tr>
<td>Expected Loss ($\gamma$)</td>
<td>1.48 E-10</td>
<td>2.24 E-11</td>
<td>-6.57</td>
</tr>
</tbody>
</table>

Year dummies are omitted. Observations: 288. Adj. $R^2 = 26\%$

Table 9: Premium Parametrizations based on NAIC reinsurance data (2002-2010).

### A.4 Implementation of the Multi-Period Model in Section 3

We rely on value iteration to solve the dynamic program (9). For solving the one-period problems in each step, we rely on the following basic algorithm. Details on some of the steps, on the implementation, and evidence on the convergence are provided below.

**Algorithm A.1.** For a given (discretized) end-of-period value function $V^{\text{end}}$, and for capital levels $a_k = ADEL \times k$, $k = 0, 1, 2, \ldots, AGRID$:

1. Given capital level $a_k$, optimize over $q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}$.

2. Given capital level $a_k$ and a portfolio $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$, optimize over $R^{(b)}$.

3. Given $a_k$, portfolio $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$, and raising decision $R^{(b)}$, determine the premium levels by evaluating Equation (10).

4. Given $a_k$, portfolio $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$, $R^{(b)}$, and premiums $(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})$ evaluate $V^{\text{beg}}(a_k)$ based on the given end-of-period function $V^{\text{end}}$ by interpolating in between the grid and extrapolating off the grid.

**Discretization**

We rely on an equidistant grid with 26 point (AGRID = 25) with different increments depending on the parameters (ADEL = 250,000,000 for the base case and ADEL = 750,000,000 for the
“profitable company” and “empty company” cases). We experimented with larger grids with finer intercepts but 26 points proved to be a suitable compromise between accuracy and run time of the program.

**Optimization**

For carrying out the numerical optimization of the portfolio values $q^{(i)}$, $i = 1, 2, 3, 4$, we rely on the so-called downhill simplex method proposed by Nelder and Mead (1965) as available within most numerical software packages. For the starting values, we rely on the optimized values from the previous step, with occasional manual adjustments during the early iteration in order to smooth out the portfolio profiles.

For the optimization of the optimal raising decision $R^b$, in order to not get stuck in a local maximum, we first calculate the value function based on sixty different values across the range of possible values $[-a \times (1 - \tau), \infty)$. We then use the optimum of these as the starting value in the Nelder-Mead method to derive the optimized value.

**Calculation of the Premium Levels**

The primary difficulty in evaluating the optimal premium level is that premiums enter the constraint (10) on both sides of the equation as the default rate itself depends on the premium, and this dependence is discontinuous (given our discrete loss distributions). We use the following approach: Starting from a zero default rate, we calculate the minimal amount necessary to attain the given default rate; we then check whether this amount is incentive-compatible, i.e., if the policyholders would be willing to pay it given the default probability. If so, we calculate a smoothed version of the premium level using (10) by deriving the (hypothetical) default rate considering how much the policyholders are willing to pay over the minimal amount at that level relative to the amount necessary to decrease the default rate based on the discrete distribution. If not, we move to the next possible default rate given our discrete loss distribution and check again.

**Interpolation and Extrapolation**

For arguments in between grid points, we use linear interpolation. For values off the grid ($a > \text{AGRID} \times \text{ADEL}$), supported by the general shape of the value functions across iterations, we use either linear or quadratic extrapolation. More precisely, in case fitting a quadratic regression in $a$ to the five greatest grid values does not yield a significant quadratic coefficient—I.e., if the value function appears linear in this region—we use linear extrapolation starting from the largest grid point. Otherwise, we use a quadratic extrapolation fitted over the entire range starting from the largest grid point.
Convergence

We assess convergence by calculating absolute and relative errors in the value and the policy functions from one iteration to the next. These errors are directly proportional to error bounds for the algorithm, where the proportionality coefficients depend on the interest and the default rate (an upper bound is given by $\bar{c} = e^{-r}/(1-e^{-r})$, see e.g. Proposition 3.1 in Bertsekas (1995, Chap. 1)). More precisely, we define the absolute and relative errors for the value function by:

$$\text{AbsErr}_n = \max_k \left\{ \| V_n(a_k) - V_{n-1}(a_k) \| \right\},$$

$$\text{RelErr}_n = \max_k \left\{ \frac{\| V_n(a_k) - V_{n-1}(a_k) \|}{V_n(a_k)} \right\},$$

where $V_n$ denotes the value function after iteration $n$. Similarly, we define absolute and relative errors for the policy function by:

$$\text{AbsErr}(q^{(i)})_n = \max_k \left\{ \| q^{(i)}_n(a_k) - q^{(i)}_{n-1}(a_k) \|, \quad i = 1, 2, 3, 4, \right\},$$

$$\text{RelErr}(q^{(i)})_n = \max_k \left\{ \frac{\| q^{(i)}_n(a_k) - q^{(i)}_{n-1}(a_k) \|}{q^{(i)}_n(a_k)} \right\}, \quad i = 1, 2, 3, 4,$$

where $q^{(i)}_n$ denotes the (optimized) exposure to line $i$ after iteration $n$.

Figure 9: Absolute and relative error in the value function for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00E-10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48E-10$ (base case).

The panels in Figure 9 show the errors for the value function using the base case parameters and different values of $n$ between 90 and 320 (in increments of 10). Error bounds for the other
parametrizations are even smaller. After 320 iterations, the absolute error in the value function is 379.230, which is only a very small fraction of the value function ranging from 1,813,454,921 to 1,955,844,603 (about 0.02%). In particular, considering the rather conservative error bound above, these results imply that the error in $V$ amounts to less than one percent. Similarly, Figure 10 shows the absolute and relative errors for the portfolio functions. Again, we observe that relative changes from one iteration to the next after 320 iterations are maximally around 0.02%.

**Remarks on the Solution**

It is interesting to note that overall, the relative portfolio allocation, $q^{(i)} / \sum_j q^{(j)}$ coincides across capital levels and even coincides between the base and the profitable company case (this was not imposed in the numerics). The reason appears to be that according to the specification of the premium function, each line exhibits the same profitability and the default states are the same across all cases. Hence, the optimal line mix is consistent and the problem amounts to choosing an optimal aggregate exposure, which is non-linear in $a$. This of course will change if we allow for differences in profitability or if it turns out to be optimal to adjust the relevant default states. In particular, in the empty company case (where optimally default no longer occurs), we obtain a different relative portfolio allocation.

**B Additional Figures**
Figure 11: Value function $V$ and its derivative $V'$ for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00E-11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61E-10$ (profitable company).

Figure 12: Optimal raising decision $R$ for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00E-11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61E-10$ (profitable company).
Figure 13: Optimal portfolio weights $q_1$, $q_2$, $q_3$, and $q_4$ for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\times10^{-11}$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\times10^{-10}$ (profitable company).

<table>
<thead>
<tr>
<th></th>
<th>$a = 0$</th>
<th>$a = 3\text{bn}$</th>
<th>$a = 12\text{bn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Value of Solvent Payments, (i)</td>
<td>$1,227,669,151$</td>
<td>$1,568,829,904$</td>
<td>$1,995,399,708$</td>
</tr>
<tr>
<td>($\mathbb{E}[I_{{I \leq D}}]$)</td>
<td>$58.03%$</td>
<td>$65.81%$</td>
<td>$77.62%$</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment, (ii)</td>
<td>$218,275,408$</td>
<td>$83,338,060$</td>
<td>$39,070,892$</td>
</tr>
<tr>
<td>($\mathbb{E}[w(I)_{{I \leq D}}]$)</td>
<td>$10.32%$</td>
<td>$3.50%$</td>
<td>$1.52%$</td>
</tr>
<tr>
<td>Capital cost, (iii)</td>
<td>$669,771,688$</td>
<td>$731,731,568$</td>
<td>$536,155,774$</td>
</tr>
<tr>
<td>($D \times \mathbb{E}[w(I)_{{I &gt; D}}]$)</td>
<td>$31.66%$</td>
<td>$30.69%$</td>
<td>$20.86%$</td>
</tr>
<tr>
<td>agg. marginal cost, (i)-(iii)</td>
<td>$2,115,716,247$</td>
<td>$2,383,899,532$</td>
<td>$2,570,626,375$</td>
</tr>
<tr>
<td></td>
<td>$100.00%$</td>
<td>$100.00%$</td>
<td>$100.00%$</td>
</tr>
</tbody>
</table>

Table 10: Total marginal cost allocation for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\times10^{-11}$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\times10^{-10}$ (profitable company).
Figure 14: Weighting function $w(I)$ for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00E-11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61E-10$ (profitable company).