Decomposing Dynamic Risks into Risk Components *

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February 2019

Abstract

The decomposition of dynamic risks a company faces into components associated with various sources of risk such as financial risks, aggregate economic risks, or industry-specific risk drivers is of significant relevance in view of risk management and product design, particularly in (life) insurance. Nevertheless, although several decomposition approaches have been proposed, no systematic analysis is available. This paper closes this gap in literature by introducing properties for *meaningful* risk decompositions and demonstrating that proposed approaches violate at least one of these properties. As an alternative, we propose a novel *MRT decomposition* that relies on martingale representation and show that it satisfies all of the properties. We discuss its calculation and present detailed examples illustrating its applicability.

Classification: 91B30, 60H07, 60J60

Keywords: risk decomposition, martingale representation, life insurance

*We are grateful for helpful comments from three expert referees and an associate editor, as well as from seminar participants at the Workshop "Perspectives on Actuarial Risks in Talks of Young Researcher" 2013, the Annual Congress of the German Insurance Science Association 2013, the 17th International Congress on Insurance: Mathematics and Economics, the 30th International Congress of Actuaries, the 2nd European Actuarial Journal Conference. An earlier version of this paper was awarded the 2015 GAUSS-Preis by the German Society for Actuarial and Financial Mathematics (DGVFM) and the German Society of Actuaries (DAV).

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1 Introduction

Decomposing risks into components associated with different sources of risk is a problem of practical significance. The primary contributions of this paper are twofold: On the one hand, we introduce properties for a *meaningful risk decomposition* and show that decomposition methods proposed in literature suffer shortcomings in view of these properties. On the other hand, we propose a novel decomposition approach based on martingale representation, labeled *MRT decomposition*, and show that it satisfies all the meaningful risk decomposition properties. We discuss and illustrate the calculation of the MRT decomposition in the context of life insurance, where it is particularly relevant.

The total risk a company faces is frequently influenced by various sources of risk such as financial risk, aggregate economic risk, and industry-specific risk factors. The interaction of these sources can be quite complex, so that the individual risk contributions are typically neither obvious nor readily available. For instance, this is the case in life insurance, where final payoffs – that commonly occur years or even decades after the origination of the contracts – depend on the interaction of financial factors and guarantees, aggregate demographic trends, and actual deaths observed in the portfolio of insured. Nonetheless, companies need to assess the relative importance of each source of risk in order to be able to devise adequate risk management strategies. This may simply be a matter of identifying the most significant source of risk for focusing efforts in case resources for risk management are limited. Alternatively, the decomposition may allow to gauge how much resources should be dedicated to each source of risk, taking into account its contribution to the aggregate risk. Evaluating the impact of different sources of risk is also important in view of product design, particularly when there are different risk penalties for different sources of risk, and in view of calculating risk-based capital requirements for financial institutions.[1]

In this paper, we commence by introducing a number of properties that define a *meaningful* risk decomposition. In particular, we posit that a decomposition should consider the entire distribution of the company's risk (P1), that resulting decompositions should be unique (P3) and independent of the ordering of the sources of risk (P4), that the different risk components can be clearly attributed to the different sources of risk (P2), that the risk components are invariant to changes in the scale of the sources of risk (P5), and, finally, that the decomposition should aggregate to the (normalized) entire risk (P6). We discuss the relevance of these properties in the context of management, concluding that all of them appear essential for an expedient decomposition. However, it turns out that when benchmarking decomposition approaches proposed in literature with this list of properties, for each method at least one of the properties fails to hold.

[1]For instance, in the insurance context, individual risk contributions need to be quantified explicitly within Solvency II, the new regulatory framework for insurance companies within the European Union (Directive 2009/138/EC, Article 112).

This leads us to propose our alternative *MRT decomposition* that is based on the Martingale Representation Theorem, hence the name. Our definition makes use of the *dynamic* nature of the considered risk, i.e. we exploit the underlying "process structure." Intuitively, the *MRT risk components* capture the non-predictable change in the total risk due to (only) the change in the corresponding risk source as we move forward an instant in time. We frame our formal introduction in a life insurance setting, where risk decompositions are particularly relevant as outlined above. More precisely, we consider a group of policyholders modeled by a counting process and we assume that the (systematic) sources of risk are driven by a finite-dimensional Brownian motion. This framing is more general than just relying on a diffusion framework as is common when (solely) considering financial risk factors, and therefore illustrates how the definition can be adapted to different settings. For instance, modeling a policyholder's time of death is similar to modeling a firm's time of default or the defect of a machine within a production company. We show that the MRT decomposition satisfies each property P1 to P6, and we document the expedience of the resulting decomposition in our specific setting. In particular, we show that the risk component associated with *unsystematic mortality risk* vanishes as the portfolio size increases – whereas the *systematic* risk components approach a non-zero limit. We derive explicit formulas for the the MRT decomposition in terms of Malliavin derivatives in the general case and in terms of derivatives of conditional expectations in the Markov case.

We illustrate the MRT decomposition in various applications. In addition to a variety of simpler motivating examples, we consider a detailed practical application in the context of a variable annuity contract with a guaranteed minimum death benefit (GMDB) – a very common product in the US insurance market. We decompose the total risk into four sources of risk: fund risk, interest rate risk, systematic mortality risk, and unsystematic mortality risk. Our calculations show that for an unhedged exposure, fund risk is by far the most dominant risk, particularly when considering moderately sized insurance portfolios. Different applications are considered in follow-up studies (Schilling, 2017; Jetses, 2018, e.g.).

Related Literature and Organization of the Paper

Dynamic risk decompositions are particularly relevant in the financial services industry, in view of risk management (Hoem, 1988), pricing (Christiansen, 2013), product design (Kochanski and Karnarski, 2011), and capital regulation. Thus, it is not surprising that there are a number of papers suggesting different methodologies for deriving risk components in the more specialized actuarial and financial risk management literatures. Bühlmann (1989), Fischer (2004), Martin and Tasche (2007), and Christiansen and Helwich (2008) use a conditional expectations approach, which is the probabilistic foundation of the well-known *variance decomposition*. In particular, Fischer (2004) provides a list of desirable properties for a reasonable decomposition method, although the focus is on the specific decomposition into financial risk and (unsystematic) mortality risk in life insurance. A generalized conditional expectations approach – the so-called Hoeffding or functional ANOVA decomposition – is used by Rosen and Saunders (2010) in the context of credit risk portfolios. Christiansen (2007) uses a generalized Taylor expansion method for decomposing functionals of different sources of risk in the insurance context. A different method, e.g. applied by Gatzert and Wesker (2014) in the context of life insurance and also implicitly used in the Solvency II framework, "switches off" the randomness of all sources of risk that are momentarily not under consideration. We revisit all of these approaches in Section 2.4 and document their shortfalls in view of our meaningful risk decomposition properties.

Decomposing a risk into its components obviously relates to studying its sensitivity with regards its risk sources, or *sensitivity analysis* in brief (see Borgonovo (2011) or Borgonovo and Plischke (2016) for surveys). More precisely, a decomposition into stochastic components is related to the problem of *global sensitivity analysis* (Saltelli et al., 2008; Wagner, 1995), where one studies the sensitivity of the model "output" to various (random) "inputs" – so that not only the sensitivity at one particular input point is relevant. For instance, there exist analogies in desirable characteristics for (global) sensitivity analysis and risk decomposition. In this context, Baucells and Borgonovo (2013) propose properties desirable to sensitivity analysis – with some similarities to our list (e.g., in view of P2 and P5) – and propose a new class of sensitivity measures on that basis. However, there are a number of aspects that differ between decomposing a risk and studying its sensitivity to risk sources. For instance, a sensitivity measure is typically a number and not a random variable (P1). Furthermore, aggregation (P6) may not be a concern – e.g., one usually relies on first-order effect functions only for assessing the "trend" identification of an input factor. Indeed, the Hoeffding decomposition, which does not satisfy P6 (see Sec. 2.4), is the foundation of much of the development of global sensitivity analysis. That said, a decomposition may serve as the basis for developing sensitivity measures. For instance, a decomposition can be used for defining *inner statistics* in the sense of Borgonovo et al. (2016), which then give rise to *global sensitivity measures*.

Risk decomposition also relates to risk capital allocation: Risk capital allocation starts with a (linearly) aggregated portfolio and determines risk contributions of the different positions via gradients of portfolio risk measures (see Bauer and Zanjani (2013) for a detailed review on capital allocation). These risk contributions, which are also referred to as *Euler allocations*, are important ingredients to the risk-adjusted return on capital (RAROC) and similar performance evaluation techniques. In fact, this practice is a key motivation for a growing literature on determining risk measure sensitivities (Fu et al., 2009; Hong, 2009; Liu, 2015, among others). A (linear) risk decomposition will allow to identify capital allocations of the different risk *components*, which provides guidance for risk management. In particular, Karabey et al. (2014) rely on several of the decomposition approaches listed above (conditional expectations, Hoeffding, and Taylor) to determine Euler allocations.

From a technical perspective, the derivation of our MRT decomposition is closely related to quadratic hedging approaches under a martingale measure (Møller, 2001; Dahl and Møller, 2006; Barbarin, 2008; Dahl et al., 2008; Biagini et al., 2013; Biagini and Schreiber, 2013; Biagini et al., 2016, among others) with the conceptual difference that we operate under the physical measure since we are interested in risk assessments. We rely on this analogy in our derivations but also present some new results in this direction such as the decomposition of arbitrary payoffs within our general setting and the integration with the Clark-Ocone formula from Malliavin calculus.

The remainder of the paper is organized as follows. Section 2 presents the properties that define a meaningful risk decomposition and analyzes whether conventional approaches from literature satisfy these properties. Section 3 lays out the considered life insurance modeling framework, introduces our MRT decomposition within this framework, and discusses its calculation and properties. Section 4 describes and analyzes our variable annuity example. And, finally, Section 5 concludes. Proofs and other technical material are provided in the E-Companion to this paper.

2 Meaningful risk decompositions

2.1 General setup

For the remainder of the paper, we fix a finite time horizon T^* and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T^*}$ satisfying the usual conditions.[2] Throughout, $\mathcal{F}_t$ describes the "full" information available at time t, where we assume $\mathcal{F}_0$ to be trivial and set $\mathcal{F} = \mathcal{F}_{T^*}$.

Let $T \in [0, T^*]$ be the maturity of the longest exposure, and let the random variable $L_{0,T}$, shortly L, denote the sum of all exposures over $[0, T]$ (possibly discounted to time 0). Then we suppose that *total risk* as from time 0 is given via the normalized random variable $R = L - \mathbb{E}(L)$, i.e. we interpret risk as the random deviation of the exposure from its expectation. Note that the discount factors possibly included in L introduce specific investment assumptions which themselves may cause risk.

The primary concern of this paper is decomposing risk R into different risk components. More precisely, we posit that there are k *sources of risk*, where each source of risk $i \in \{1, 2, \ldots, k\}$ is modeled by an $\mathbb{F}$-adapted stochastic process $Z_i = (Z_i(t))_{0\leq t\leq T^*}$, and we write $Z = (Z(t))_{0\leq t\leq T^*}$ with $Z(t) = (Z_1(t), \ldots, Z_k(t))^\top$. The risk variable R is assumed to be $\sigma(Z)$-measurable, i.e. it

[2] In principle, $\mathbb{P}$ could be any probability measure and the technical results in this paper are not fixed to a particular interpretation. However, since our focus is on a company's risk, we interpret $\mathbb{P}$ as the real-word measure and this interpretation is reflected in our language.

(only) depends on the information spanned by Z. Then we consider decomposition methodologies that assign each source of risk Z_i a corresponding risk component R_i, which itself is a $\sigma(Z)$-measurable random variable and which is supposed to capture the randomness of R caused by Z_i. Here the maximal exposure time T is fixed, so that R is not time-dependent.

Hence, it is important to note that we consider a *static* decomposition problem although the underlying setting is *dynamic*. Indeed, we could also define the *meaningful risk decompositions* in the next subsection within a simpler, static setting, but the novel MRT decomposition method we propose exploits the dynamic nature.

2.2 Definition of meaningful risk decompositions

While several papers propose a variety of decomposition methods in a similar context, thus far there has been no systematic assessment and comparison among these different approaches. In what follows, we introduce a list of properties we argue a *meaningful risk decomposition* should satisfy. In particular, minimum requirements for the relation between the given sources of risk $Z_1, \dots, Z_k$ and the corresponding risk components $R_1, \dots, R_k$ resulting from the decompositions are postulated (equalities between random variables are in the almost sure sense):

P1 *Randomness*

Individual risk components are given by exactly k (possibly degenerate) *random variables* $R_1,\ R_2, \dots,\ R_k$. We introduce the relation $\leftrightarrow$ for a decomposition methodology and write $(R, Z_1, \dots, Z_k) \leftrightarrow (R_1, R_2, \dots, R_k)$ to indicate that the risk R depending on $(Z_1, \dots, Z_k)$ corresponds to the decomposition $(R_1, R_2, \dots, R_k)$.

P2 *Attribution*

R_i represents the risk component related to source of risk i. Formally, we require that whenever the risk R is $\sigma(Z_i)$-measurable and Z_i is stochastically independent of $(Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_k)$, then $R_j = 0$ for all $j \neq i$.

P3 *Uniqueness*

The decomposition methodology yields a unique decomposition. Formally, we require that $(R, Z_1, \dots, Z_k) \leftrightarrow (R_1, R_2, \dots, R_k)$ and $(R, Z_1, \dots, Z_k) \leftrightarrow (\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_k)$ implies $R_i = \tilde{R}_i$ for all $i \in \{1, 2, \dots, k\}$.

P4 *Order invariance*

The decomposition is invariant to the order of the sources of risk $1,\ 2, \dots,\ k$. Formally, consider a permutation $\pi : \{1, 2, \dots, k\} \to \{1, 2, \dots, k\}$ and assume $(R, Z_1, \dots, Z_k) \leftrightarrow$

$(R_1, R_2, \dots, R_k)$. Then we require:

$$(R, Z_{\pi(1)}, \dots, Z_{\pi(k)}) \leftrightarrow (R_{\pi(1)}, R_{\pi(2)}, \dots, R_{\pi(k)}).$$

P5 *Scale invariance*

The decomposition is invariant to changes in the scale of the sources of risk. Formally, assume $(R, Z_1, \dots, Z_k) \leftrightarrow (R_1, R_2, \dots, R_k)$, and let $\tilde{Z}_i(t) = f_i(Z_i(t))$ for all $i = 1, \dots, k,\ t \in [0, T^*]$, where, for each i, $f_i : \mathbb{R} \to \mathbb{R}$ is a smooth, invertible function. If $(R, \tilde{Z}_1, \dots, \tilde{Z}_k) \leftrightarrow (\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_k)$, then we require that $R_i = \tilde{R}_i$ for all $i \in \{1, \dots, k\}$.

P6 *Aggregation*

The decomposition aggregates to the total risk faced by the company. Formally, we require that for each risk R and sources of risk Z with $(R, Z_1, \dots, Z_k) \leftrightarrow (R_1, R_2, \dots, R_k)$, there exists a function $A_{(R,Z)} : \mathbb{R}^k \to \mathbb{R}$ such that:

$$R = A_{(R,Z)}(R_1, R_2, \dots, R_k).$$

P6* *Additive aggregation*

A special case of P6 is an additive aggregation function, i.e. the case where R is given as the sum of the individual risk components:

$$R = \sum_{i=1}^{k} R_i.$$

Note that the relation $\leftrightarrow$ will be a function if P3 is satisfied. Furthermore, if additionally P6 holds and the function $A_{(R,Z)}$ does not depend on R (as is e.g. the case under P6*), then $\leftrightarrow$ is injective in R for fixed Z since:

$$(R_1, \dots, R_k) = (\tilde{R}_1, \dots, \tilde{R}_k) \Rightarrow R = A_Z(R_1, \dots, R_k) = A_Z(\tilde{R}_1, \dots, \tilde{R}_k) = \tilde{R}.$$

2.3 Discussion Part 1: Why should a risk decomposition be meaningful?

Randomness (P1) and Aggregation (P6) are not only desirable characteristics. Rather, they describe the nature of a risk decomposition. Namely, they postulate that we can *decompose* the total liability risk into k fragments from which the total risk can be retrieved. Obviously aggregation would not be possible if the individual components were not the same elements as the total risk – i.e. if they were not random variables. The latter point deserves emphasis: As opposed to sensitivity measures or risk (capital) allocations – which are numeric – a risk decomposition reflects the full randomness of the components.

Attribution (P2), Uniqueness (P3), and Order Invariance (P4) establish a clear link between a risk source and the resulting component. Such a clear link is important when the decomposition drives managerial decisions, since it would be problematic when spurious or immaterial aspects have material influence. More precisely, absence of uniqueness (P3) implies that depending on some exogenous influence (e.g., a parameter), the decomposition will be $(R_1, R_2, \dots, R_k)$ or $(\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_k)$ leaving the manager with choosing the "right" one. Similarly, it should not matter of whether a risk source that is completely irrelevant to the problem is included in the scope, which is what P2 expresses. Finally, the simple label of what the "first" risk source is should not matter – a mere relabeling of the risk sources should not affect the decomposition, which is what P4 entails.

Scale invariance (P5) is necessary to ensure that the risk components are quantitatively comparable. Scaling the risk sources via a given transformation does not change the nature of the risk, and thus should not affect decompositions (consider e.g. a change in currency or a change to logarithmic scale for one of the risk sources). In particular, without P5, there may exist incentives to adjust decompositions via the framing of the problem, which is not desirable if they influence decision making.

An additive decomposition (P6*) is desirable for multiple reasons. It allows for the natural interpretation that the risk components *add up* to the total risk. Moreover, for a decomposition into summands, it is straightforward to derive capital allocations via the gradients of risk measures.

2.4 Discussion Part 2: Are conventional approaches meaningful?

For discussing conventional decomposition approaches with regards to whether or not they satisfy the meaningful risk decomposition properties, for simplicity we assume $T^* = T = 1$ and that the random variable L is only influenced by two sources of risk $Z_1 = (Z_1(t))_{0\le t\le 1}$ and $Z_2 = (Z_2(t))_{0\le t\le 1}$. In particular, we set:

$$Z_1(t) = \theta_1\, t + \sigma_1\, W_1(t) \text{ and } Z_2(t) = \theta_2\, t + \sigma_2\, (\rho\, W_1(t) + \sqrt{1-\rho^2}\, W_2(t)),$$

where $W_1 = (W_1(t))_{0\le t\le 1}$ and $W_2 = (W_2(t))_{0\le t\le 1}$ are independent Brownian motions, $\theta_i \in \mathbb{R}$, $\sigma_i \ge 0$, and $0 \le \rho \le 1$, $i \in \{1,2\}$. Hence, $Z_1(1) \sim N(\theta_1, \sigma_1^2)$, $Z_2(1) \sim N(\theta_2, \sigma_2^2)$, and $\text{Correl}(Z_1(1), Z_2(1)) = \rho$.

Recall that total risk is identified with $R = L - \mathbb{E}(L)$. To preview our results, we find that each considered risk decomposition approach fails to satisfy at least one property, whereas the MRT decomposition that we formally introduce in the following section is *meaningful*.

Variance decomposition

A common approach for decomposing R into risk components is a conditional expectations approach (Bühlmann, 1989; Fischer, 2004; Martin and Tasche, 2007; Christiansen and Helwich, 2008, among others). The basic idea is that $R_1 = \mathbb{E}\left(R \middle| Z_1 \right)$ captures the randomness of R caused by Z_1. Since the remaining risk $R_2 = R - R_1 = R - \mathbb{E}\left(R \middle| Z_1 \right)$ must represent the randomness caused by Z_2, the decomposition of the risk $R = L - \mathbb{E}(L)$ reads as:

$$R = \mathbb{E}\left(R \middle| Z_1 \right) + \left[R - \mathbb{E}\left(R \middle| Z_1 \right) \right] = \underbrace{\left[\mathbb{E}\left(L \middle| Z_1 \right) - \mathbb{E}(L) \right]}_{=R_1} + \underbrace{\left[L - \mathbb{E}\left(L \middle| Z_1 \right) \right]}_{=R_2}, \tag{1}$$

with R_1 and R_2 representing the two risk components.

As a result of the orthogonality property of conditional expectations, a straightforward consequence of (1) is:

$$\mathrm{Var}(R) = \mathrm{Var}(R_1) + \mathrm{Var}(R_2).$$

Commonly, the latter equation is referred to as *variance decomposition* and frequently it is the basis for applications (thus, we simply refer to the general decomposition (1) as "variance decomposition"). Note that for an arbitrary risk R, the variance decomposition directly implies that $\mathbb{E}(R_1) = \mathbb{E}(R)$ and $\mathbb{E}(R_2) = 0$. Of course, this asymmetry is irrelevant when considering the variance but potentially relevant when applying different risk measures. This emphasizes the necessity to first standardize L to mean zero, i.e. considering the risk $R = L - \mathbb{E}(L)$, and to then apply the decomposition approach, resulting in $\mathbb{E}(R_1) = \mathbb{E}(R_2) = 0$. We note that in global sensitivity analysis, ${}^{\mathrm{Var}(R_1)}/_{\mathrm{Var}(R)}$ is referred to as the *first-order sensitivity index* of Z_1 on R (Saltelli et al., 2008, Eq. (1.25)), illustrating the relationship to risk decomposition.

Obviously, the risk components R_1 and R_2 are random variables (P1) and they add up to the total risk (P6*/P6). Since conditional expectations are unique almost surely, so is the variance decomposition (P3). To check the attribution property (P2), for independent processes Z_1 and Z_2 and a $\sigma(Z_1)$-measurable risk R, $R_2 = R - \mathbb{E}\left(R \middle| Z_1 \right) = 0$. Conversely, if R is $\sigma(Z_2)$-measurable, $R_1 = \mathbb{E}\left(R \middle| Z_1 \right) = \mathbb{E}(R)$. Therefore, P2 is satisfied since R is standardized to mean zero. The variance decomposition is also scale invariant (P5), since for two smooth, invertible functions f_1 and f_2, with $\tilde{Z}_i(t) = f_i(Z_i(t))$, $i = 1, 2$, we have $\tilde{R}_1 = \mathbb{E}(R| \tilde{Z}_1) = \mathbb{E}(R| Z_1) = R_1$ and $\tilde{R}_2 = R - \mathbb{E}(R| \tilde{Z}_1) = R - \mathbb{E}(R| Z_1) = R_2$. However, as the following example illustrates, the order invariance property (P4) is not satisfied:

Example 1 *(Cf. Exercise 1.7 in Saltelli et al. (2008)). Assume that* $\rho = 0$ *and* $L = Z_1(1)\, Z_2(1)$*.*

Then the variance decomposition of $R = L - \mathbb{E}(L)$ with respect to $Z = (Z_1, Z_2)$ is given by:

$$\begin{aligned} R &= \mathbb{E}\left(Z_2(1)\right)\left[Z_1(1) - \mathbb{E}\left(Z_1(1)\right)\right] + Z_1(1)[Z_2(1) - \mathbb{E}\left(Z_2(1)\right)] \\ &= \underbrace{\theta_2\,\sigma_1\,W_1(1)}_{=R_1} + \underbrace{\left(\theta_1 + \sigma_1\,W_1(1)\right)\sigma_2\,W_2(1)}_{=R_2}. \end{aligned}$$

In contrast, switching the order of Z_1 and Z_2, i.e. considering $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) = (Z_2, Z_1)$, the variance decomposition approach yields:

$$R = \underbrace{\theta_1\,\sigma_2\,W_2(1)}_{=\tilde{R}_1} + \underbrace{\left(\theta_2 + \sigma_2\,W_2(1)\right)\sigma_1\,W_1(1)}_{=\tilde{R}_2}.$$

Clearly, in general $R_1 \neq \tilde{R}_2$ and $R_2 \neq \tilde{R}_1$. In particular, if $\theta_1 = \theta_2 = 0$, the first decomposition will imply $R_1 = 0$ and $R_2 = R$ whereas the second decomposition will yield $\tilde{R}_1 = 0$ and $\tilde{R}_2 = R$, i.e. either no risk or the total risk will be attributed to Z_1 (or vice versa for Z_2). □

In addition, although Z_1 and Z_2 may be correlated, R_1 and R_2 will be uncorrelated. This means that correlated risk must be allocated in an uncorrelated way, which can further result in arbitrary, order-dependent decompositions:

Example 2 *Consider $L = Z_1(1) + Z_2(1)$ with $\rho > 0$. Then the variance decomposition of $R = L - \mathbb{E}(L)$ with respect to $Z = (Z_1, Z_2)$ is given by:*

$$\begin{aligned} R &= \left[Z_1(1) - \mathbb{E}\left(Z_1(1)\right) + \mathbb{E}\left(Z_2(1)|\,Z_1\right) - \mathbb{E}\left(Z_2(1)\right)\right] + \left[Z_2(1) - \mathbb{E}\left(Z_2(1)|\,Z_1\right)\right] \\ &= \underbrace{\left(\sigma_1 + \sigma_2\,\rho\right)\,W_1(1)}_{=R_1} + \underbrace{\sigma_2\,\sqrt{1-\rho^2}\,W_2(1)}_{=R_2}. \end{aligned}$$

In contrast, switching the order of Z_1 and Z_2, i.e. considering $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) = (Z_2, Z_1)$, the variance decomposition approach yields:

$$R = \underbrace{\left(\sigma_2 + \sigma_1\,\rho\right)\,\left(\rho\,W_1(1) + \sqrt{1-\rho^2}\,W_2(1)\right)}_{=\tilde{R}_1} + \underbrace{\sigma_1\,(1-\rho^2)\,W_1(1) - \sigma_1\,\rho\,\sqrt{1-\rho^2}\,W_2(1)}_{=\tilde{R}_2},$$

where again $R_1 \neq \tilde{R}_2$ and $R_2 \neq \tilde{R}_1$ unless $\rho = 0$. □

Hoeffding decomposition

A related approach is based on the so-called *Hoeffding* or *functional ANOVA* decomposition (Hoeffding, 1948; Efron and Stein, 1981; Sobol, 1993). See, for instance, Rosen and Saunders (2010) or Karabey et al. (2014) for applications to credit risk and life insurance, respectively.

Similarly to the previous approach, it relies on conditional expectations. For the risk $R = L - \mathbb{E}(L)$, the Hoeffding decomposition reads:

$$
\begin{aligned}
R &= \mathbb{E}\left(R|\, Z_1\right) + \mathbb{E}\left(R|\, Z_2\right) + \left[R - \mathbb{E}\left(R|\, Z_1\right) - \mathbb{E}\left(R|\, Z_2\right)\right] \\
&= \underbrace{\mathbb{E}\left(L|\, Z_1\right) - \mathbb{E}\left(L\right)}_{=R_1} + \underbrace{\mathbb{E}\left(L|\, Z_2\right) - \mathbb{E}\left(L\right)}_{=R_2} + \underbrace{L - \mathbb{E}\left(L|\, Z_1\right) - \mathbb{E}\left(L|\, Z_2\right) + \mathbb{E}\left(L\right)}_{=R_{1,2}},
\end{aligned}
$$

where R_1 and R_2 are the risk components attributed to Z_1 and Z_2 in isolation, and $R_{1,2}$ represents the risk due to "joint effects." This immediately illustrates the decomposition's primary drawback, namely that the total risk is not completely allocated to the individual sources of risk. In order to have exactly one risk component for each risk source (P1), one possibility is to ignore the joint term $R_{1,2}$ of the Hoeffding decomposition and only consider the individual risk components following the so-called Hájek projection, $R \approx R_1 + R_2$ (Hájek, 1968). However, as the following example shows, this leads to immediate problems with the meaningful decomposition properties, particularly the aggregation property (P6).

Example 1 (Continued) *Consider $L = Z_1(1)\, Z_2(1)$ and assume both risks have mean zero, $\theta_1 = \theta_2 = 0$. Then the Hoeffding approach yields $R_1 = R_2 = 0$ and $R_{1,2} = R$, i.e. the total risk results from joint effects, which does not give any insights on the influence of the different sources of risk. In particular, this example shows that the aggregation property (P6) is generally not satisfied since for every function $A_{(R,Z)} : \mathbb{R}^2 \to \mathbb{R}$ we have $A_{(R,Z)}(R_1, R_2) = A_{(R,Z)}(0, 0) \neq R$ whenever L is not deterministic.* □

However, properties P1 to P5 are satisfied for the Hájek projection: Clearly, the risk components R_1 and R_2 are random variables (P1). Scale invariance (P5) follows by the same argument as for the variance decomposition. And for the attribution property (P2), let Z_1 and Z_2 be two independent processes; if R is $\sigma(Z_1)$-measurable, then R is independent of Z_2, so that $R_2 = \mathbb{E}\left(R|\, Z_2\right) = \mathbb{E}\left(R\right) = 0$ (analogously for $\sigma(Z_2)$-measurable R). Furthermore, the primary effects of the decomposition R_1 and R_2 are unique by the uniqueness of the conditional expectations (P3) and obviously order invariant (P4), although dependence in the risk sources can generate spurious higher-order terms in the Hoeffding decomposition (Saltelli and Tarantola, 2002; Borgonovo and Plischke, 2016).

Taylor expansion

Christiansen (2007) proposes approximating functionals of random variables by their first order Taylor expansion and interprets the resulting summands as risk components. More precisely, assume L is of the form $F(Z_1(1), Z_2(1))$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a Borel-measurable and differentiable function. Then this approach yields:

$$R = L - \mathbb{E}(L) \approx [F(z_1, z_2) - \mathbb{E}(L)] + \underbrace{\frac{\partial F}{\partial z_1}(z_1, z_2)(Z_1(1) - z_1)}_{=R_1} + \underbrace{\frac{\partial F}{\partial z_2}(z_1, z_2)(Z_2(1) - z_2)}_{=R_2},$$

where (z_1, z_2) denotes the (deterministic) expansion point. By using a generalized definition of the corresponding gradients, Christiansen (2007) extends this approach to an infinite-dimensional setting where L may depend on the entire paths of the processes Z_1 and Z_2.

As the key drawback, in case of non-linear functionals the first-order Taylor expansion and its summands only approximate the risk R. In particular, the approximation error with a certain realization highly depends on the choice of the expansion point, i.e. the Taylor expansion is "local," echoing similar limitations of traditional/local sensitivity analysis relative to *global* sensitivity analysis (Saltelli et al., 2008; Borgonovo and Plischke, 2016).

Example 1 (Continued) *The Taylor expansion with expansion point (z_1, z_2) yields:*

$$\begin{aligned} R = L - \mathbb{E}(L) &\approx [z_1 z_2 - \mathbb{E}(L)] + \underbrace{z_2(Z_1(1) - z_1)}_{=R_1} + \underbrace{z_1(Z_2(1) - z_2)}_{=R_2} \\ &= L - \mathbb{E}(L) - (Z_1(1) - z_1)(Z_2(1) - z_2). \end{aligned}$$

Obviously, the approximation error amounts to $-(Z_1(1) - z_1)(Z_2(1) - z_2)$, i.e. the more $Z_1(1)$ and $Z_2(1)$ deviate from z_1 and z_2, respectively, the higher is the approximation error. In the special case of choosing $(z_1, z_2) = (0, 0)$ as expansion point, the decomposition results in $R_1 = R_2 = 0$, i.e. a risk is neither allocated to Z_1 nor to Z_2. As a result, the aggregation property (P6) generally is not satisfied since for every function $A_{(R,Z)} : \mathbb{R}^2 \to \mathbb{R}$ we have $A_{(R,Z)}(R_1, R_2) = A_{(R,Z)}(0, 0) \neq R$ (assuming that $Z_1(1)\, Z_2(1)$ is not deterministic). Furthermore, due to the dependence on the expansion point, the Taylor expansion approach is also not unique (P3). $\square$

To show that scale invariance (P5) is violated, consider the following example.

Example 3 *Assume that $L = e^{Z_1(1)}$. Then the Taylor expansion yields:*

$$R \approx e^{z_1} - \mathbb{E}\left(e^{Z_1(1)}\right) + e^{z_1}(Z_1(1) - z_1)$$

for some expansion point z_1. However, for $\tilde{Z}_1(1) = e^{Z_1(1)}$ and $\tilde{z}_1 = e^{z_1}$ we have:

$$R \approx \tilde{z}_1 - \mathbb{E}\big(\tilde{Z}_1(1)\big) + (\tilde{Z}_1(1) - \tilde{z}_1),$$

and in general $R_1 = e^{z_1}(Z_1(1) - z_1) \neq e^{Z_1(1)} - e^{z_1} = \tilde{Z}_1(1) - \tilde{z}_1 = \tilde{R}_1$. $\square$

Still, the Taylor expansion satisfies properties P1, P2, and P4 (at least for the specific functionals). The risk components are obviously random variables, and order invariance can be easily shown. For the attribution property, independence of the other source of risk will yield a zero derivative and thus a zero risk component.

OAT approach

A different risk decomposition relies on "switching off" all of the randomness and then analyzing the sources one-at-a-time (OAT). In the insurance domain, in addition to applications in research (Gatzert and Wesker, 2014, e.g.), this method is in principle implied in the Solvency II framework for measuring the influence of different sources of risk (CEIOPS, 2010). Moreover, it is related to basic one-at-a-time (OAT) or one-factor-at-a-time (OFAT) approaches in (local) sensitivity analysis (Borgonovo, 2011; Borgonovo and Plischke, 2016).

To illustrate, assume again that L is of the form $F(Z_1(1), Z_2(1))$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a Borel-measurable function. Then the method suggests to model the risk components of $R = L - \mathbb{E}(L)$ corresponding to Z_1 and Z_2 via $R_1 = F(Z_1(1), z_2) - \mathbb{E}(L)$ and $R_2 = F(z_1, Z_2(1)) - \mathbb{E}(L)$, respectively, where $z_1, z_2 \in \mathbb{R}$. In the context of Solvency II, z_1 and z_2 are typically chosen as best estimates of $Z_1(1)$ and $Z_2(1)$. However, in general there is no clear answer on how z_1 and z_2 should be chosen (again in analogy to the distinction of local vs. global sensitivity analysis). In fact, the decomposition heavily depends on the choice of z_1 and z_2 and is thus not unique (P3). Generally, the risk components also do not aggregate (P6). Both points are illustrated in the following example.

Example 4 *Assume that $L = Z_1(1) \max\{K - Z_2(1), 0\}$, where K is a constant with $\mathbb{E}(Z_2(1)) = \theta_2 > K$. Measuring the risk component of $R = L - \mathbb{E}(L)$ related to Z_1 by replacing $Z_2(1)$ with its expectation, the OAT approach yields:*

$$R_1 = Z_1(1) \max\{K - \theta_2, 0\} - \mathbb{E}(L) = -\mathbb{E}(L).$$

Thus, although $R > 0$ with positive probability (assuming that L is non-constant) and R is increasing in the realization of $Z_1(1)$, the risk attributed to Z_1 is constant and possibly even negative. However, choosing any deterministic approximation $z_2 < K$ yields $R_1 = Z_1(1)(K - z_2) - \mathbb{E}(L)$ with a different distribution for each choice of z_2. Beyond uniqueness, the OAT approach also does not satisfy the aggregation property P6 (and thus also not P6*), which follows immediately from the above with $z_2 = \mathbb{E}(Z_2(1)) = \theta_2 > K$: for any function $A_{(R,Z)} : \mathbb{R}^2 \to \mathbb{R}$, $A_{(R,Z)}(R_1, R_2)$ will be $\sigma(Z_2)$-measurable and thus $A_{(R,Z)}(R_1, R_2) \neq R$ (assuming that Z_1 is not $\sigma(Z_2)$-measurable).
□

Summary of Decomposition Approaches

	P1	P2	P3	P4	P5	P6	P6*
Variance decomposition	✓	✓	✓	×	✓	✓	✓
Hoeffding decomposition	✓	✓	✓	✓	✓	×	×
Taylor expansion	✓	✓	×	✓	×	×	×
OAT approach	✓	×	×	✓	✓	×	×
MRT decomposition	✓	✓	✓	✓	✓	✓	✓

Table 1: Summary of decomposition approaches with regards to whether (✓) or not (×) they satisfy the properties P1 to P6*.

Furthermore, the attribution property is generally not satisfied (P2):

Example 3 (Continued) *For* $L = e^{Z_1(1)} = F(Z_1(1), Z_2(1))$, R *is* $\sigma(Z_1)$*-measurable, but for every* $z_1 \neq \log(\mathbb{E}(L))$*, we have* $R_2 = F(z_1, Z_2(1)) - \mathbb{E}(L) = e^{z_1} - \mathbb{E}(L) \neq 0$. □

In contrast, the OAT approach satisfies properties P1, P4, and P5 (at least for the specific functionals). Again, the risk components are obviously random variables (P1), and order invariance can be easily shown (P4). For scale invariance (P5), let f_1 and f_2 be two smooth, invertible functions and define $\tilde{Z}_i(1) = f_i(Z_i(1))$ and $\tilde{z}_i = f_i(z_i),\ i = 1, 2$. It directly follows that $L = F(Z_1(1), Z_2(1)) = F(f_1^{-1}(\tilde{Z}_1(1)), f_2^{-1}(\tilde{Z}_2(1))) = \tilde{F}(\tilde{Z}_1(1), \tilde{Z}_2(1))$. Hence, we have $\tilde{R}_1 = \tilde{F}(\tilde{Z}_1(1), \tilde{z}_2) - \mathbb{E}(L) = F(Z_1(1), z_2) - \mathbb{E}(L) = R_1$ and analogously for $\tilde{R}_2 = R_2$ (assuming the change of scale is the same for $\tilde{z}_i$ as for $\tilde{Z}_i(1),\ i = 1, 2$).

Table 1 summarizes the results. For each decomposition, at least one of the properties fails to hold. Motivated by these deficiencies, we introduce a novel decomposition approach labeled *MRT decomposition* that satisfies all *meaningful risk decomposition* properties. In what follows, we illustrate and motivate the MRT decomposition in the context of the examples above, before we provide a more formal introduction, discussion, and application in the context of life insurance in the following sections.

MRT Decomposition (informal)

We revisit the examples from above using the MRT decomposition. We refer to the E-Companion, Part B, for details on calculation using the tools we provide in the following sections.

Example 1 (Continued) *The MRT decomposition yields:*

$$\begin{aligned} R &= Z_1(1)\, Z_2(1) - \mathbb{E}\left(Z_1(1)\, Z_2(1)\right) \\ &= \underbrace{\int_0^1 (Z_2(t) + \theta_2(1-t))\, \sigma_1\, dW_1(t)}_{=R_1} + \underbrace{\int_0^1 (Z_1(t) + \theta_1(1-t))\, \sigma_2\, d\left(\rho\, W_1(t) + \sqrt{1-\rho^2}\, W_2(t)\right)}_{=R_2}. \end{aligned}$$

The example illustrates the key intuition behind the MRT decomposition: The MRT risk components capture the non-predictable change in the total risk due to (only) the change in the corresponding risk source as we move forward an instant in time. In particular, for risk source one, the non-predictable change at time t is given by innovations in the underlying Brownian motion W_1, $dW_1(t) = W_1(t+dt) - W_1(t)$. This change is scaled by σ_1 and $(Z_2(t) + \mathbb{E}[Z_2(1) - Z_2(t)])$ from the definition of Z_1 and R, respectively, and aggregated through time to give R_1 (and analogously for R_2). In particular, this demonstrates how our definition makes use of the dynamic nature of considered risk. We exploit the underlying "process structure" to improve on the approaches above that rely on static decompositions. The components R_1 and R_2 are random (P1), attributable (P2), unique (P3), order-independent (P4), scale-invariant (P5), and add-up to the total risk (P6).* □

Example 2 (Continued) *For $L = Z_1(1) + Z_2(1)$, the MRT decomposition yields:*

$$\begin{aligned} R &= Z_1(1) + Z_2(1) - \mathbb{E}\left(Z_1(1) + Z_2(1)\right) \\ &= \underbrace{\int_0^1 \sigma_1\, dW_1(t)}_{=R_1} + \underbrace{\int_0^1 \sigma_2\, d(\rho\, W_1(t) + \sqrt{1-\rho^2}\, W_2(t))}_{=R_2} \\ &= \overbrace{Z_1(1) - \mathbb{E}\left(Z_1(1)\right)} + \overbrace{Z_2(1) - \mathbb{E}\left(Z_2(1)\right)}. \qquad \square \end{aligned}$$

Example 3 (Continued) *For $L = e^{Z_1(1)} = \tilde{Z}_1(1)$, the MRT decomposition yields:*

$$\begin{aligned} R = e^{Z_1(1)} - \mathbb{E}\left(e^{Z_1(1)}\right) &= \underbrace{\int_0^1 e^{Z_1(t)}\, e^{(\theta_1 + \sigma_1^2/2)\,(1-t)}\, \sigma_1\, dW_1(t)}_{=R_1} = R \\ &= \overbrace{\int_0^1 \tilde{Z}_1(t)\; e^{(\theta_1 + \sigma_1^2/2)\,(1-t)}\, \sigma_1\, dW_1(t)} = R. \qquad \square \end{aligned}$$

Examples 2 and 3 demonstrate that in addition to – or, rather, because of – satisfying the meaningful decomposition properties, the resulting components are in line with what intuition suggests. For additive risks (in the risk sources), the MRT decomposition simply yields the corresponding summands, irrespective of their ordering (P4) and their correlation. And for a risk that is only influenced by a single source of risk, the component will equate with the risk itself (P2), irrespective of scaling (P5).

However, as the following example shows, the MRT decomposition also delivers in settings where it is more difficult to develop an intuition.

Example 4 (Continued) *To simplify exposition, we assume* $\theta_1 = \theta_2 = \rho = 0$. *Then the MRT decomposition for* $L = Z_1(1)\max\{K - Z_2(1), 0\}$ *yields:*

$$\begin{aligned} R &= L - \mathbb{E}(L) \\ &= \underbrace{\int_0^1 \left((K - Z_2(t))\, \Phi\left(\frac{K - Z_2(t)}{\sigma_2 \sqrt{1-t}}\right) + \sigma_2 \frac{\sqrt{1-t}}{\sqrt{2\pi}} \exp\left\{ -\frac{(K - Z_2(t))^2}{2\,\sigma_2^2\,(1-t)} \right\} \right) dW_1(t)}_{=R_1} \\ &\quad + \underbrace{\int_0^1 Z_1(t)\, \Phi\left(\frac{K - Z_2(t)}{\sigma_2 \sqrt{1-t}}\right) \sigma_2\, (-dW_2(t))}_{=R_2}, \end{aligned}$$

where Φ *and* ϕ *are the standard normal cumulative and density distribution functions, respectively. The second component* R_2 *may be more intuitive: The instantaneous risk in* R_2 *is modulated by* $Z_1(t)$ *due to the factor structure of* L*, it is relevant with time-t probability* $\mathbb{P}_t(K - Z_2(1) > 0) = \Phi\left(\frac{K - Z_2(t)}{\sigma_2 \sqrt{1-t}}\right)$*, and an increase in* W_2 *decreases* L *so that the last term carries a minus sign. The first part of the first component* R_1 *has a similar interpretation. To obtain intuition for the second part, note that even for* $Z_2(t) = K$ *the risk in* $Z_1(t)$ *is non-zero – even though the first part is – since it is possible that* $Z_2(t) < K$ *an instant later. In fact,* σ_2 *measures how fast* $Z_2(t)$ *will move away from* K*. This second part assures that the risk components add up to the total risk (P6**). $\square$

3 MRT decomposition in life insurance

We frame our formal introduction of the MRT decomposition in a life insurance setting. The reasons are twofold. On the one hand, as detailed above, risk decompositions are particularly relevant in life insurance, in view of risk management, pricing, product design, and capital regulation. Indeed, applications in life insurance were the initial motivation for this research. On the other hand, this framing is more general than simply considering a diffusion framework – as is popular when solely considering financial risks – and therefore illustrates how the definition can be adapted to other situations in insurance and beyond. In particular, all the examples considered in the previous section also fall within our framework. The first part lays out the setting for the remainder of the paper. Section 3.2 introduces the MRT decomposition and shows that it is *meaningful* in the sense of Section 2.2. Section 3.3 discusses its calculation, where we rely on analogies to hedging problems for insurance liabilities. And, finally, we analyze diversification properties in Section 3.4.

3.1 Life insurance modeling framework

On the basis of the general setup described in Section 2.1, we specify the sources of risk $Z_1, \dots, Z_k$ as well as L as they are typical for life insurance. We assume that the uncertainty of the insurer's future loss L arises from the uncertain evolution of a number of financial and demographic factors as well as the actual occurrence of deaths in the insurance portfolio. For the former, we introduce an n-dimensional, locally bounded process $X = ((X_1(t), \dots, X_n(t))^\top)_{0 \le t \le T^*}$, the so-called *state process*, where each process X_i is interpreted as source of risk, and assume that all financial and demographic factors are functions of X. Specifically, we assume that the time-t prices of all risky assets on the financial market as well as the interest rate $r(t) = r(t, X(t))$ can be expressed in terms of $X(t)$.[3] The state process itself is driven by a d-dimensional standard Brownian motion $W = ((W_1(t), \dots, W_d(t))^\top)_{0 \le t \le T^*}$, where $\mathbb{G}$ denotes the augmented filtration generated by W, which is assumed to be a sub-filtration of $\mathbb{F}$:

Assumption 1 *The state process $X = ((X_1(t), \dots, X_n(t))^\top)_{0 \le t \le T^*}$ is an n-dimensional Itô process satisfying:*

$$dX(t) = \theta(t)\, dt + \sigma(t)\, dW(t) \tag{2}$$

with deterministic initial value $X(0) = x_0 \in \mathbb{R}^n$, where the n-dimensional drift vector $\theta = (\theta(t))_{0 \le t \le T^*}$ and the $n \times d$-dimensional volatility matrix $\sigma = (\sigma(t))_{0 \le t \le T^*}$ are $\mathbb{G}$-adapted with continuous paths.

For notational convenience and without much loss of generality, we consider m homogeneous policyholders aged a at time 0. As is conventional in settings with aggregate demographic uncertainty (Biffis, 2005; Biffis et al., 2010; Dahl and Møller, 2006; Dahl et al., 2008, among many others), we model the remaining lifetime τ_a^i of policyholder i as seen from time 0, $i = 1, \dots, m$, as the first jump time of a *doubly stochastic* or *Cox process* with intensity $(\mu(t))_{0 \le t \le T^*}$, where $\mu(t) = \mu(t, X(t))$ is non-negative and continuous. That is, the probability to decease in the next instant is contingent on aggregate stochastic demographic factors that are included in the state process, whereas the event of death is triggered by an idiosyncratic random jump. As pointed out in Biffis et al. (2010), we can construct – and simulate – individual death times via:

$$\tau_a^i = \inf \left\{ t \in [0, T^*] : \int_0^t \mu(s) ds \ge E_i \right\}, \quad i = 1, \dots, m,$$

where E_i, $i = 1, \dots, m$, are i.i.d. unit exponential random variables independent of $\mathcal{G}_{T^*}$ and where we use the convention $\inf \varnothing = \infty$ for individuals that survive beyond the time horizon

[3] Within particular models, the prices of risky assets, interest rates, or mortality indices may be components of the state process X.

$(\tau_a^i > T^*)$. In particular, the residual lifetimes $\tau_a^i,\ i = 1, \ldots, m$, of the homogeneous policyholders are by construction conditionally identically distributed and conditionally independent given the σ-algebra $\mathcal{G}_{T^*}$. Defining $\mathbb{I} = \bigvee_{i=1}^m \mathbb{I}^i$, where $\mathbb{I}^i = (\mathcal{I}_t^i)_{0\le t\le T^*}$ is the augmented filtration generated by the death indicator process $(\mathbf{1}_{\{\tau_a^i \le t\}})_{0\le t\le T^*}$, we naturally assume that $\mathbb{F}$ in Section 2.1 is given by $\mathbb{F} = \mathbb{G} \vee \mathbb{I}$. Writing $\Gamma(t) = \int_0^t \mu(s)ds$ for the so-called cumulative mortality intensity, we immediately obtain:

Lemma 1 *For* $t, s \in [0, T^*],\ t \le s,\ i = 1, \ldots, m$*:*

1. $\mathbb{P}\left(\tau_a^i > t | \mathcal{G}_t\right) = e^{-\int_0^t \mu(s)ds} = e^{-\Gamma(t)}$ *and particularly* $\mathbb{P}\left(\tau_a^i > 0\right) = 1$*;*
2. $\mathbb{P}\left(\tau_a^i > t | \, \mathcal{G}_{T^*}\right) = \mathbb{P}\left(\tau_a^i > t | \, \mathcal{G}_s\right)$*;*[4] *and*
3. $\mathbb{P}\left(\tau_a^i > T | \mathcal{F}_t\right) = \mathbb{P}\left(\tau_a^i > T | \mathcal{G}_t \vee \mathcal{I}_t^i\right) = \mathbf{1}_{\{\tau_a^i > t\}} \dfrac{\mathbb{P}\left(\tau_a^i > T | \mathcal{G}_t\right)}{\mathbb{P}\left(\tau_a^i > t | \mathcal{G}_t\right)} = \mathbf{1}_{\{\tau_a^i > t\}} \mathbb{E}\left(e^{-\int_t^T \mu(s)ds} \Big| \, \mathcal{G}_t \right)$.

Generalizations of the setting that preserve these results are possible (Jeanblanc and Rutkowski, 2000; Biffis et al., 2010).

Each life insurance contract in the company's portfolio is assumed to entail the same cash flows, the only difference being the remaining lifetimes. Thus, we denote the number of policyholders that have died until time t by $N(t) = \sum_{i=1}^m \mathbf{1}_{\{\tau_a^i \le t\}}$, so that $N = (N(t))_{0\le t\le T^*}$ represents the only further source of risk in addition to the sources of risk $X_i, i = 1, \ldots, n$. Let $\mathbb{F}^{W,N} = \left(\mathcal{F}_t^{W,N}\right)_{0\le t\le T^*}$ denote the augmentation of the filtration generated by the processes W and N and note that $\mathbb{F}^{W,N}$ is a sub-filtration of $\mathbb{F}$.

For motivating the considered sources of risk and, in particular, for calculating the proposed decomposition later on, we need to specify the insurer's loss L. To keep the setting general, the cash flows associated with each life insurance contract in the company's portfolio may be independent of the lifetimes, contingent on the policyholder's survival, or contingent on the policyholder's death. Furthermore, we distinguish between discrete as well as continuous cash flows. Each cash flow may include several payments from and to the insurance company. Positive payments are interpreted as payments made by the insurer (mostly benefits paid) and negative payments are interpreted as payments received by the insurer (mostly premiums). Consequently, we define the insurer's total loss L as:

$$L = C_{0,T} + \sum_{j=0}^{\ell} (m - N(t_j))C_{a,j} + \int_0^T (m - N(t))C_a(t)\, dt + \int_0^T C_{ad}(t)\, dN(t), \qquad (3)$$

where $0 = t_0 < t_1 < \ldots < t_\ell = T,\ \ell \in \mathbb{N}$, are discrete points in time. In detail, the different quantities in (3) denote:

[4] According to Jeanblanc and Rutkowski (2000), this is equivalent to the so-called $\mathcal{H}$-hypothesis, which says that every $\mathbb{G}$-martingale remains a martingale with respect to the larger filtration $\mathbb{F}$.

- $C_{0,T}$: The sum of all (possibly discounted) payments in $[0, T]$ that do not depend on $\tau_a^i,\ i = 1, \ldots, m$, such as hedging returns, benefits from a fixed-term insurance, etc.;
- $C_{a,j}$: The sum of all (possibly discounted) payments at or after time t_j that are contingent on survival until time t_j, $j = 0, \ldots, \ell$, such as single premiums, discrete premium payments, discrete annuity payments, benefits from pure endowment insurances, benefits from period certain deferred annuities, etc. as well as death benefits paid (see below);
- $C_a(t)$: time-t intensity of all continuous payments that are contingent on survival until time t or, in other words, $C_a(t)\,dt$ is the sum of all payments in the infinitesimal period $[t, t{+}dt]$ that are contingent on survival until time $t,\ t \in [0, T]$, such as continuous premium payments, continuous annuity payments, etc.;
- $C_{ad}(t)$: the sum of all (possibly discounted) payments at time t that are contingent on death at time $t,\ t \in [0, T]$, such as death benefits paid immediately upon death.

We do not explicitly include discrete payments contingent on death within $(t_{j-1}, t_j]$, $j = 1, \ldots, l$, say $C_{ad,j}$, such as death benefits paid at the end of the period since:

$$(N(t_j) - N(t_{j-1}))\, C_{ad,j} = (m - N(t_{j-1}))C_{ad,j} - (m - N(t_j))C_{ad,j} \tag{4}$$

can be represented as a difference of two discrete survival cash flows. For this reason, we assume that $C_{0,T}$ and $C_{a,j}$ are $\mathcal{G}_{T^*}$-measurable, i.e. the cash flows $C_{a,j}$ may depend on market information after time t_j, whereas $C_a(t)$ and $C_{ad}(t)$ are assumed to be $\mathcal{G}_t$-measurable. This assumption also allows us to include unit-linked annuity contracts with a guaranteed annuity period and similar policies: For instance, with a five year guarantee period starting at age 65, a payment at age 68 may depend on the state of the market *then* but may be solely contingent on survival past 65. Indeed, this distinction in measurability is one of the main reasons why we explicitly include continuous as well as discrete cash flows. Note that due to its form, the total loss L is $\mathcal{F}^{W,N}_{T^*}$-measurable.

3.2 The MRT decomposition

Within the life insurance modeling framework introduced in the previous section, the objective is to find an approach that decomposes the insurer's risk $R = L - \mathbb{E}(L)$ into risk components attributed to the sources of risk the insurer faces in a *meaningful* way (cf. Section 2.2). Inspired by the martingale representation theorem, we propose a decomposition into stochastic integrals with respect to the *compensated* sources of risk and interpret each integral as the risk component of the respective source of risk. Here "compensated" means that as for the risk itself, we subtract the predictable part ("trend") from each of the sources of risk. These subtracted terms are then referred to as the *compensators*. This is necessary since the expected value of the risk R – which is zero after the normalization – needs to match up with the decomposition.

As introduced in the previous section, the sources of risk are identified with, on the one hand, the state processes $X_i = (X_i(t))_{0 \leq t \leq T^*},\ i = 1, \ldots, n$, and, on the other hand, with the number of deaths in the portfolio $N = (N(t))_{0 \leq t \leq T^*}$. The corresponding compensated processes, i.e. the processes less their $\mathbb{F}$-compensators, are denoted by $M_i^W = (M_i^W(t))_{0 \leq t \leq T^*},\ i = 1, \ldots, n$, and $M^N = (M^N(t))_{0 \leq t \leq T^*}$, respectively. The MRT decomposition is then defined as follows:

Definition 1 *Let L be $\mathcal{F}_{T^*}^{W,N}$-measurable and $R = L - \mathbb{E}(L)$. A decomposition of the form:*

$$R = \sum_{i=1}^{n} \int_0^{T^*} \psi_i^W(t)\, dM_i^W(t) + \int_0^{T^*} \psi^N(t)\, dM^N(t), \tag{5}$$

where $\psi_i^W(t),\ i = 1 \ldots, n$, and $\psi^N(t)$ are $\mathbb{F}$-predictable processes, is called MRT decomposition of R. The corresponding MRT risk components are given as:

$$R_i = \int_0^{T^*} \psi_i^W(t) dM_i^W(t),\ i = 1, \ldots, n,\ \textit{and}\ R_{n+1} = \int_0^{T^*} \psi^N(t) dM^N(t).$$

We write $(R, X_1, \ldots, X_n, N) \overset{MRT}{\leftrightarrow} (R_1, \ldots, R_{n+1})$.

Obviously, each integral in (5) is interpreted as the portion of the total randomness of R caused by the associated source of risk.[5]

Remark 1 *Our definition focuses on the insurer's risk as from time 0. However, with appropriate modifications all related definitions and results can be transferred to the insurer's risk at any future time $t \in [0, T^*]$ (considered from time 0). For example, in analogy to L and Equation* (3)*, the insurer's total loss at time t, $L_{t,T}$, can be defined as the sum of all future cash flows in $[t, T]$, where possible discount factors of the cash flows need to be adjusted to time t. The insurer's risk at time t then follows as $L_{t,T} - \mathbb{E}\left(L_{t,T} \middle| \mathcal{F}_t \right)$. A decomposition of the form* (5) *can be analogously found for $L_{t,T} - \mathbb{E}\left(L_{t,T}\right)$ as for $L - \mathbb{E}\left(L\right)$ (with possibly different integrands). Since all integrals in* (5) *are martingales, it follows for the insurer's risk at time t that (for simplicity, we use the same notation for the integrands as above):*

$$L_{t,T} - \mathbb{E}\left(L_{t,T} \middle| \mathcal{F}_t \right) = \sum_{i=1}^{n} \int_t^{T^*} \psi_i^W(s) dM_i^W(s) + \int_t^{T^*} \psi^N(s) dM^N(s),$$

[5] Similar interpretations of stochastic integrals can be found e.g. in Christiansen (2013) for unsystematic risk and in Biagini et al. (2013) under a risk-neutral measure. In particular, the risk component R_{n+1} describes the randomness introduced by N, i.e. by the random occurrence of deaths in the portfolio, and thus corresponds to the inherent *unsystematic mortality risk*.

and the corresponding MRT risk components can be defined as integrals starting from t.

The motivation for and the MRT decomposition's existence and uniqueness are implied by the martingale representation theorem, as shown by the proposition below. However, we first provide the specification of the compensated processes:

Lemma 2 *1. The unique compensator of X_i is given by $A_i^W = (A_i^W(t))_{0\le t\le T^*}$, where $A_i^W(t) = \int_0^t \theta_i(s)\,ds,\ i = 1,\dots,n$. Thus:*

$$M_i^W(t) = \sum_{j=1}^{d} \int_0^t \sigma_{ij}(s)\,dW_j(s),\ \ 0 \le t \le T^*,\ i = 1,\dots,n.$$

2. The unique compensator of N is given by $A^N = (A^N(t))_{0\le t\le T^*}$, where $A^N(t) = \int_0^t (m - N(s-))\,\mu(s)\,ds$. Thus:

$$M^N(t) = N(t) - \int_0^t (m - N(s-))\,\mu(s)\,ds,\ \ 0 \le t \le T^*,$$

where, for completeness, we define $N(0-) = 0$.

Proposition 1 *Assume that $n = d$, $\det\{\sigma(t)\} \ne 0$ for all $t \in [0,T^*]$ $\mathbb{P}$-almost surely, and that L is $\mathcal{F}_{T^*}^{W,N}$-measurable and square integrable. Then there exist $\mathbb{F}^{W,N}$-predictable processes $\psi_1^W,\dots,\psi_n^W,\psi^N : [0,T^*]\times\Omega \to \mathbb{R}$ such that the MRT decomposition (5) of $R = L - \mathbb{E}(L)$ exists. The representation is unique in the sense that the integrands $\psi_1^W,\dots,\psi_n^W$ and the integrand ψ^N are a.s. unique on $[0,T^*]\times\Omega$ and $\{(t,\omega)\in[0,T^*]\times\Omega : N(t-) < m\}$, respectively, both with respect to $\lambda\otimes\mathbb{P}$, where λ denotes the Lebesgue measure on $[0,T^*]$. Moreover:*

$$\mathbb{E}\left(\left[\int_0^{T^*} \psi^N(t)dM^N(t)\right]^2\right) < \infty. \tag{6}$$

Remark 2 *The previous proposition is based on the assumption that each insurance contract in the considered portfolio entails the same cash flows. For relaxing this assumption, it is sufficient to split the considered portfolio into sub-portfolios with identical cash flows and to apply the result from above to each sub-portfolio separately. Moreover, if the payments depend on the sequence of deaths as within joint life policies, it is possible to extend the setting and consider the processes in the general filtration $\mathbb{F}$ implying $d + m$ driving martingales. We focus on $\mathbb{F}^{W,N}$ here since it is the most relevant setup and to keep the presentation concise.*

If $n \ne d$, existence and uniqueness of the MRT decomposition (5) are not necessarily given. In fact, as follows from the proof, we need to look for $\psi^W(t)$ such that the equation $\widetilde{\psi}^W(t) =$

$\psi^W(t)\sigma(t)$ holds true, where existence and uniqueness of $\widetilde{\psi}^W(t)$ result from the martingale representation theorem. If $n > d$, there are fewer equations than unknowns so that uniqueness is not guaranteed. On the other hand, if $n < d$, there are more equations than unknowns so that a solution may not exist. In what follows, we focus on the case $n = d$. If $n \neq d$, we assume that either redundant state processes (which can be represented via other state processes) are removed or additional state processes are artificially added, both along with an adjustment of the interpretation of the risk sources. In contrast to a hedging problem, where the number of state processes – or rather securities – is exogenously given, this procedure is viable for a risk decomposition problem.

The key motivation for the definition of the MRT decomposition is provided by the following proposition: It satisfies the meaningful risk decomposition properties from Section 2.2.

Proposition 2 *Assume that the state process $X = (X_1, \ldots, X_n)$ is defined as in Assumption 1 with $n = d$ and $\det\{\sigma(t)\} \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Let L be $\mathcal{F}^{W,N}_{T^*}$-measurable and square integrable. Then the MRT decomposition:*

$$(R, X_1, \ldots, X_n, N) \overset{MRT}{\leftrightarrow} (R_1, \ldots, R_{n+1})$$

defined via (5) *satisfies the properties P1, P2, P3, P4, P5, P6, and P6*.*

Remark 3 *While the notion of uniqueness of the MRT decomposition (P3) is based on the description of X in Assumption 1, it is important to note that it will not depend on the representation of X, since the compensated risk processes M^W_i, $i = 1, \ldots, n$, coincide for each representation. In particular, we will obtain the same MRT decomposition if we choose a representation of X in terms of correlated Brownian motions.*

3.3 Calculation of the MRT decomposition

The calculation of the MRT decomposition amounts to determining the integrands $\widetilde{\psi}^W_1, \ldots, \widetilde{\psi}^W_n, \psi^N$ in (5). The fundamental theorem of calculus states that integrands in Riemann integrals are given by a function's derivatives. In analogy, the integrands here are given by certain derivatives, although these derivatives have to be taken with regards to random variables/processes. This is the subject of the so-called *Malliavin calculus* or *stochastic calculus of variations* (see Nualart (2006) and Di Nunno et al. (2009) for detailed introductions). Specifically, denoting by $\mathbb{D}_{1,2}$ the set of random variables that are Malliavin differentiable with respect to each one-dimensional Brownian motion W_i of $W = (W_1, \ldots, W_d)$, the time-t Malliavin derivative with respect to W_i, $D_{t,i}(\cdot)$, satisfies:

$$D_{t,i}\left(\sum_{j=1}^{d} \int_0^T f_j(s)\, dW_j(s)\right) = f_i(t),\ t \in [0, T],\ i = 1, \ldots, d,$$

where $\sum_{j=1}^{d}\int_0^T f_j(s)\,dW_j(s) \in \mathbb{D}_{1,2}$.

A relatively general result for the calculation of $D_{t,i}$ is the Clark-Ocone formula (Di Nunno et al., 2009, Section 4, e.g.). However, the Clark-Ocone formula is only applicable to independent driving processes, which renders a direct application to our setting impossible. More precisely, since we are considering stochastic mortality intensities, the number of deaths in the portfolio N and the standard Brownian motion W driving, among others, the mortality intensity will generally not be independent. Thus, we rely on the specific structure of the loss random variable – and particularly the separation into discrete survival cash flows, continuous survival cash flows, and continuous cash flows contingent on death – to reduce the problem to finding the martingale representation of a $\mathbb{G}$-martingale instead of an $\mathbb{F}$-martingale. That is, we reduce the problem to a diffusion setting, and we can then apply the Clark-Ocone formula to obtain the MRT decomposition of each summand of L defined in Equation (3) – and thus the MRT decomposition of L itself by summing up the individual decompositions.

Proposition 3 *Assume that $n = d$ and that the inverse $\sigma^{-1}(t) = \left(\sigma_{ij}^{-1}(t)\right)_{i,j=1,\dots,n}$ exists for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Let $0 \le t_k \le T^*,\ 0 \le T \le T^*$.*

1. ***(Financial cash flow)*** *Let $L = C_{0,T}$. If $C_{0,T} \in \mathbb{D}_{1,2}$, then the unique integrands of the MRT decomposition* (5) *of $R = L - \mathbb{E}(L)$ are given by:*

$$\psi_i^W(t) = \sum_{j=1}^{d} \mathbb{E}\left(D_{t,j}\left(C_{0,T}\right)\middle|\, \mathcal{G}_t\right)\sigma_{ji}^{-1}(t),\ \ i = 1,\dots,n,\ \psi^N(t) = 0.$$

2. ***(Discrete survival cash flow)*** *Let $L = (m - N(t_k))C_{a,k}$. If $e^{-\Gamma(t_k)}C_{a,k} \in \mathbb{D}_{1,2}$, then the unique integrands of the MRT decomposition* (5) *of $R = L - \mathbb{E}(L)$ are given by:*

$$\begin{aligned}\psi_i^W(t) = &\left[(m - N(t-))e^{\Gamma(t)}\mathbf{1}_{[0,t_k]}(t) + (m - N(t_k))e^{\Gamma(t_k)}\mathbf{1}_{(t_k,T^*]}(t)\right]\\ &\times \sum_{j=1}^{d}\mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\middle|\,\mathcal{G}_t\right)\sigma_{ji}^{-1}(t),\ \ i = 1,\dots,n,\\ \psi^N(t) = &-\mathbf{1}_{[0,t_k]}(t)\,\mathbb{E}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\,\mathcal{G}_t\right).\end{aligned}$$

3. ***(Continuous survival cash flow)*** *Let $L = \int_0^T (m - N(t))C_a(t)dt$. If $C_a = (C_a(t))_{0\le t\le T}$ is a $\mathbb{G}$-predictable process with $\mathbb{E}\left(\sup_{t\in[0,T]}|C_a(t)|\right) < \infty$, $\sup_{t\in[0,T]}\mathbb{E}\left([C_a(t)]^2\right) < \infty$, and $e^{-\Gamma(t)}C_a(t) \in \mathbb{D}_{1,2}$ for all $t \in [0, T]$, then the unique integrands of the MRT decomposition*

(5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\psi_i^W(t) = \mathbf{1}_{[0,T]}(t)\,(m - N(t-))e^{\Gamma(t)} \times \sum_{j=1}^{d} \int_t^T \mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(v)}C_a(v)\right)\big|\,\mathcal{G}_t\right) dv\,\sigma_{ji}^{-1}(t),$$
$$i = 1, \ldots, n,$$
$$\psi^N(t) = -\mathbf{1}_{[0,T]}(t) \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)}C_a(v)\big|\mathcal{G}_t\right) dv.$$

4. ***(Continuous cash flow contingent on death)*** *Let* $L = \int_0^T C_{ad}(t)dN(t)$. *If* $C_{ad} = (C_{ad}(t))_{0\le t\le T}$ *is a continuous and* $\mathbb{G}$*-predictable process with* $\mathbb{E}\left(\sup_{t\in[0,T]}|C_{ad}(t)|\right) < \infty$, $\sup_{t\in[0,T]}\mathbb{E}\left([C_{ad}(t)]^4\right) < \infty$, $\sup_{t\in[0,T]}\mathbb{E}\left(\mu^4(t)\right) < \infty$, *and* $e^{-\Gamma(t)}C_{ad}(t)\mu(t) \in \mathbb{D}_{1,2}$ *for all* $t \in [0,T]$, *then the unique integrands of the MRT decomposition* (5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\psi_i^W(t) = \mathbf{1}_{[0,T]}(t)\,(m - N(t-))e^{\Gamma(t)} \times \sum_{j=1}^{d} \int_t^T \mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(v)}C_{ad}(v)\mu(v)\right)\big|\,\mathcal{G}_t\right) dv\,\sigma_{ij}^{-1}(t),$$
$$i = 1, \ldots, n,$$
$$\psi^N(t) = -\mathbf{1}_{[0,T]}(t)\left[\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)}C_{ad}(v)\mu(v)\big|\,\mathcal{G}_t\right) dv - C_{ad}(t)\right].$$

Remark 4 *A very similar problem arises in the context of (quadratic) hedging of insurance liabilities, and a number of papers have taken a similar approach (Møller, 2001; Dahl and Møller, 2006; Barbarin, 2008; Dahl et al., 2008; Biagini et al., 2013; Biagini and Schreiber, 2013; Biagini et al., 2016). We heavily rely on this line of research but present several extensions and new adaptations. We refer to the E-Companion, Lemmas 3, 4, and 5 in the proof of Proposition 3 as well as the corresponding Remark 6, for details on our technical contribution in this context.*

In the important special case where the state process X is a Markov process and where insurance payments are functions of the state variables, we are able to provide a more refined representation. More precisely, we can directly evaluate the decompositions via Itô's lemma rather than relying on Malliavin derivatives as in Proposition 3.

Proposition 4 *Assume that* $n = d$ *and that the state process* $X = ((X_1(t), \ldots, X_n(t))^\top)_{0\le t\le T^*}$ *is an* n*-dimensional diffusion process satisfying:*

$$dX(t) = \theta(t, X(t))dt + \sigma(t, X(t))dW(t) \tag{7}$$

with deterministic initial value $X(0) = x_0 \in \mathbb{R}^n$*, where the drift vector* $\theta : [0,T^*] \times \mathbb{R}^n \to \mathbb{R}^n$ *and the volatility matrix* $\sigma : [0,T^*] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *are continuous functions such that a unique strong solution to* (7) *exists and* $\det\{\sigma(t,X(t))\} \neq 0$ *for all* $t \in [0,T^*]$ $\mathbb{P}$*-almost surely. Furthermore, let* $T \in [0,T^*]$ *and let* $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *with* $\int_0^T |g(s,X(s))|\,ds < \infty$ $\mathbb{P}$*-almost surely and* $h : \mathbb{R}^n \to [0,\infty)$ *be some measurable functions.*[6]

1. ***(Financial cash flow)*** *Let* $L = C_{0,T}$*. Assume that* $C_{0,T}$ *is square integrable and of the form* $C_{0,T} = e^{-\int_0^T g(s,X(s))ds} h(X(T))$*. If* $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $(t,x) \mapsto \mathbb{E}\left(e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right)$*, is in* $C^{1,2}((0,T)\times\mathbb{R}^n)$*, then the unique integrands of the MRT decomposition* (5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\psi_i^W(t) = \mathbf{1}_{[0,T]}(t)\, e^{-\int_0^t g(s,X(s))ds} \frac{\partial f}{\partial x_i}(t,X(t)),\ \ i = 1,\dots,n,\ \psi^N(t) = 0.$$

2. ***(Discrete survival cash flow)*** *Let* $L = (m - N(t_k))C_{a,k},\ t_k \in [0,T^*]$*. Assume that* $C_{a,k}$ *is square integrable and of the form* $C_{a,k} = e^{-\int_0^T g(s,X(s))ds} h(X(T))$*. Let* $t_{min} = \min\{t_k, T\}$ *and define:*

$$\begin{aligned}
&f^A : [0,t_{min}] \times \mathbb{R}^n \to \mathbb{R},\ (t,x) \mapsto \mathbb{E}\left(e^{-\int_t^{t_k} \mu(s,X(s))ds} e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right),\\
&f^B : [0,T] \times \mathbb{R}^n \to \mathbb{R},\ (t,x) \mapsto \mathbb{E}\left(e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right),\\
&f^C : [0,t_k] \times \mathbb{R}^n \to \mathbb{R},\ (t,x) \mapsto \mathbb{E}\left(e^{-\int_t^{t_k} \mu(s,X(s))ds} \middle| X(t) = x\right).
\end{aligned}$$

   If $f^A \in C^{1,2}((0,t_{min}) \times \mathbb{R}^n)$ *and if in case* $t_k < T$ *additionally* $f^B \in C^{1,2}((0,T)\times\mathbb{R}^n)$ *and in case* $t_k > T$ *additionally* $f^C \in C^{1,2}((0,t_k)\times\mathbb{R}^n)$*, then the unique integrands of the MRT decomposition* (5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\begin{aligned}
\psi_i^W(t) &= \mathbf{1}_{[0,t_k]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds} \frac{\partial f^A}{\partial x_i}(t,X(t))\\
&\quad + \mathbf{1}_{(t_{min},T]}(t)\,(m - N(t_k))e^{-\int_0^t g(s,X(s))ds} \frac{\partial f^B}{\partial x_i}(t,X(t))\\
&\quad + \mathbf{1}_{(t_{min},t_k]}(t)\,(m - N(t-))C_{a,k} \frac{\partial f^C}{\partial x_i}(t,X(t)),\ \ i = 1,\dots,n,\\
\psi^N(t) &= -\mathbf{1}_{[0,T]}(t)\, e^{-\int_0^t g(s,X(s))ds} f^A(t,X(t)) - \mathbf{1}_{(t_{min},t_k]}\, C_{a,k}\, f^B(t,X(t)).
\end{aligned}$$

3. ***(Continuous survival cash flow)*** *Let* $L = \int_0^T (m - N(t))C_a(t)dt$*. Assume that* $C_a(t)$ *is of the form* $C_a(t) = e^{-\int_0^t g(s,X(s))ds} h(X(t))$ *with* $\mathbb{E}\left(\sup_{t\in[0,T]} |C_a(t)|\right) <$

[6] In what follows, we write $f \in C^{1,2}((0,T)\times\mathbb{R}^n)$ for a function $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ if the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $1 \le i,j \le n$, exist and are continuous on $(0,T)\times\mathbb{R}^n$.

∞ *and* $\sup_{t\in[0,T]} \mathbb{E}\left([C_a(t)]^2\right) < \infty$. *If* $f^v : [0,v] \times \mathbb{R}^n \to \mathbb{R}$, $(t,x) \mapsto \mathbb{E}\left(e^{-\int_t^v [g(s,X(s))+\mu(s,X(s))]ds} h(X(v)) \middle| X(t) = x\right)$, *is in* $C^{1,2}((0,v)\times\mathbb{R}^n)$ *for all* $v \in [0,T]$, *then the unique integrands of the MRT decomposition* (5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\psi_i^W(t) = \mathbf{1}_{[0,T]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds} \int_t^T \frac{\partial f^v}{\partial x_i}(t, X(t))dv, \;\; i = 1, \ldots, n,$$
$$\psi^N(t) = -\mathbf{1}_{[0,T]}(t)\, e^{-\int_0^t g(s,X(s))ds} \int_t^T f^v(t, X(t))dv.$$

4. ***(Continuous cash flow contingent on death)*** *Let* $L = \int_0^T C_{ad}(t)dN(t)$. *Assume that* $(C_{ad}(t))_{0\le t\le T}$ *is a continuous process of the form* $C_{ad}(t) = e^{-\int_0^t g(s,X(s))ds} h(X(t))$ *with* $\mathbb{E}\left(\sup_{t\in[0,T]} |C_{ad}(t)|\right) < \infty$, $\sup_{t\in[0,T]} \mathbb{E}\left([C_{ad}(t)]^4\right) < \infty$, $\sup_{t\in[0,T]} \mathbb{E}\left(\mu^4(t)\right) < \infty$. *If* $f^v : [0,v] \times \mathbb{R}^n \to \mathbb{R}$, $(t,x) \mapsto \mathbb{E}\left(e^{-\int_t^v [g(s,X(s))+\mu(s,X(s))]ds} h(X(v))\mu(v, X(v)) \middle| X(t) = x\right)$, *is in* $C^{1,2}((0,v)\times\mathbb{R}^n)$ *for all* $v \in [0,T]$, *then the unique integrands of the MRT decomposition* (5) *of* $R = L - \mathbb{E}(L)$ *are given by:*

$$\psi_i^W(t) = \mathbf{1}_{[0,T]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds} \int_t^T \frac{\partial f^v}{\partial x_i}(t, X(t))dv, \;\; i = 1, \ldots, n,$$
$$\psi^N(t) = -\mathbf{1}_{[0,T]}(t)\,\left[e^{-\int_0^t g(s,X(s))ds} \int_t^T f^v(t, X(t))dv - C_{ad}(t)\right].$$

To illustrate the above propositions and their relationship, we determine the MRT decomposition of a pure endowment portfolio in the following example.

Example 5 *Consider a portfolio of* m *pure endowment policies with survival benefit* 1 *at time* T *and single premium* P_0 *at time* 0. *For simplicity, assume a zero interest rate, so that the insurer's time-*0 *loss equals* $L = -mP_0 + (m - N(T))$. *The mortality intensity is assumed to be a non-negative affine diffusion process:*

$$d\mu(t) = \theta(t, \mu(t))\,dt + \sigma(t, \mu(t))\,dW(t),\; \mu(0) = \mu_0,$$

where $(W(t))_{0\le t\le T^*}$ *is a one-dimensional standard Brownian motion, so that (Biffis, 2005):*

$$\mathbb{E}\left(e^{-\int_t^T \mu(s)ds} \middle| \mathcal{G}_t\right) = e^{\alpha(t)+\beta(t)\mu(t)},\; T \in (t, T^*], \tag{8}$$

where α *and* β *satisfy certain Riccati ordinary differential equations.*

By applying part 2. of Proposition 3 to $(m - N(T))$ *it follows that:*

$$\begin{aligned} R &= L - \mathbb{E}(L) \\ &= \int_0^T (m - N(t-))e^{\Gamma(t)} \frac{\mathbb{E}\left(D_t \left(e^{-\Gamma(T)} \right) \middle| \mathcal{G}_t \right)}{\sigma(t, \mu(t))} \, dM^W(t) - \int_0^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(T)} \middle| \mathcal{G}_t \right) \, dM^N(t) \\ &= \int_0^T (m - N(t-)) \, e^{\alpha(t)+\beta(t)\mu(t)} \beta(t) \, dM^W(t) - \int_0^T e^{\alpha(t)+\beta(t)\mu(t)} \, dM^N(t), \end{aligned}$$

where we use the chain rule and the exchangeability of the conditional expectation and Malliavin derivative operator for evaluating $\mathbb{E}\left(D_t \left(e^{-\Gamma(T)} \right) \middle| \mathcal{G}_t \right)$ *(see the E-Companion, Part B, for details).*

Alternatively, we can apply Proposition 4, part 2. Obviously, the mortality intensity in this setting is a one-dimensional diffusion process, and we have that $C_{a,1} = e^{-\int_0^T g(s,X(s))ds} h(X(T))$ *for* $g \equiv 0$ *and* $h \equiv 1$, *where* $t_0 = 0$ *and* $t_1 = T$. *The affine property of the mortality model yields:*

$$\mathbb{E}\left(e^{-\int_t^T [\mu(s,X(s))+g(s,X(s))]ds} h(X(T)) \middle| \mathcal{G}_t \right) = \mathbb{E}\left(e^{-\int_t^T \mu(s)ds} \middle| \mathcal{G}_t \right) = e^{\alpha(t)+\beta(t)\mu(t)} = f^A(t, \mu(t)),$$

so that again:

$$R = L - \mathbb{E}(L) = \int_0^T (m - N(t-)) \, e^{\alpha(t)+\beta(t)\mu(t)} \beta(t) \, dM^W(t) - \int_0^T e^{\alpha(t)+\beta(t)\mu(t)} \, dM^N(t).$$

Note that here the first summand represents the systematic mortality risk and the second summand the unsystematic mortality risk. In the next section, we show that the latter part vanishes as the portfolio size grows.

3.4 Diversification properties

It is well known that unsystematic mortality risk arising from finite insurance portfolios vanishes as the number of policyholders goes to infinity, i.e. it is *diversifiable*. In the next proposition, we show that the risk component associated with unsystematic mortality risk within the MRT decomposition also satisfies this property. On the one hand, this corroborates the adequacy of the MRT decomposition, and, on the other hand, it allows for a crisp definition of unsystematic (mortality) risk within an insurance payoff.

Proposition 5 *Assume the setting and assumptions from Proposition 3 with resulting unsystematic* *mortality risks in part 2., 3., and 4. of, respectively:*

$$R_{n+1,ak}^{(m)} = -\int_0^{t_k} \mathbb{E}\left(e^{\Gamma(t)-\Gamma(t_k)} C_{a,k} \middle| \mathcal{G}_t \right) dM^N(t),$$

$$R_{n+1,a}^{(m)} = -\int_0^T \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)} C_a(s) \middle| \mathcal{G}_t \right) ds\, dM^N(t),$$

$$R_{n+1,ad}^{(m)} = -\int_0^T \int_t^T \left[\mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)} C_{ad}(s)\mu(s) \middle| \mathcal{G}_t \right) ds - C_{ad}(t) \right] dM^N(t).$$

1. *If* $C_{a,k} \in L^4(\mathbb{P})$ *and* $\sup_{t\in[0,t_k]} \mathbb{E}\left(\mu^2(t)\right) < \infty$, *then* $\frac{1}{m} R_{n+1,ak}^{(m)} \xrightarrow[m\to\infty]{L^2} 0$.

2. *If* $\sup_{t\in[0,T]} \mathbb{E}\left(\mu^2(t)\right) < \infty$ *and* C_a *is bounded, then* $\frac{1}{m} R_{n+1,a}^{(m)} \xrightarrow[m\to\infty]{L^2} 0$.

3. *If* C_{ad} *is bounded, then* $\frac{1}{m} R_{n+1,ad}^{(m)} \xrightarrow[m\to\infty]{L^2} 0$.

While unsystematic mortality risk diversifies, Proposition 6 shows that the remaining risk components also converge with the number of contracts, but in general not to zero, i.e. they are *not diversifiable*. This confirms their interpretation as systematic risks, particularly since the limits no longer depend on $N(t)$. In applications, if the portfolio is sufficiently large, the limits can be used as risk approximations.

Proposition 6 *Assume the setting and assumptions from Proposition 3 with resulting systematic risks in part 2., 3., and 4. of:*

$$R_{i,\cdot}^{(m)} = \int_0^T \left[(m - N(t-))e^{\Gamma(t)} \mathbf{1}_{[0,t_k]}(t) + (m - N(t_k))e^{\Gamma(t_k)} \mathbf{1}_{(t_k,T]}(t) \right] \sum_{j=1}^d \varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\, dM_i^W(t),$$

where $0 \le T \le T^*$, *and for the different parts:*

$$\varphi_{j,ak}(t) = \mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(t_k)} C_{a,k} \right) \middle| \mathcal{G}_t \right) \quad \textit{(part 2.)},$$

$$\varphi_{j,a}(t) = \int_t^T \mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(s)} C_a(s) \right) \middle| \mathcal{G}_t \right) ds \quad \textit{(part 3., where } t_k = T\textit{)},$$

$$\varphi_{j,ad}(t) = \int_t^T \mathbb{E}\left(D_{t,j}\left(e^{-\Gamma(s)} C_{ad}(s)\mu(s) \right) \middle| \mathcal{G}_t \right) ds \quad \textit{(part 4., where } t_k = T\textit{)}.$$

Then it follows for $i = 1, \ldots, n$*:*

$$\frac{1}{m} R_{i,\cdot}^{(m)} \xrightarrow[m\to\infty]{P} \int_0^T \sum_{j=1}^d \varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t) dM_i^W(t).$$

The following corollary emphasizes that the limits of the considered risk components exactly equal the risk components of the limit of the corresponding total risk, i.e. MRT decomposition and limit can be interchanged.

Corollary 1 *Assume the setting and assumptions from Proposition 3 with total risks in part 2., 3., and 4. of, respectively:*

$$L_{ak}^{(m)} = (m - N(t_k))C_{a,k}, \;\; L_a^{(m)} = \int_0^T (m - N(t))C_a(t)dt, \;\; L_{ad}^{(m)} = \int_0^T C_{ad}(t)dN(t).$$

Then the following holds:

1. $\frac{1}{m} L_{\cdot}^{(m)} \xrightarrow[m\to\infty]{a.s.} \mathbb{E}\left(L_{\cdot}^{(1)} \middle| \mathcal{G}_{T^*} \right).$
2. *Defining the MRT decompositions*
   - $(L_{\cdot}^{(m)} - \mathbb{E}(L_{\cdot}^{(m)}), X_1, \ldots, X_n, N) \overset{MRT}{\leftrightarrow} (R_{1,\cdot}^{(m)}, \ldots, R_{n+1,\cdot}^{(m)}), \; m \in \mathbb{N}$, *and*
   - $(\mathbb{E}(L_{\cdot}^{(1)} | \mathcal{G}_{T^*}) - \mathbb{E}(L_{\cdot}^{(1)}), X_1, \ldots, X_n, N) \overset{MRT}{\leftrightarrow} (R_{1,\cdot}^{*}, \ldots, R_{n+1,\cdot}^{*}),$

   and additionally assuming the respective assumptions of Proposition 5, then it follows for $i = 1, \ldots, n+1$*:*

$$\frac{1}{m} R_{i,\cdot}^{(m)} \xrightarrow[m\to\infty]{P} R_{i,\cdot}^{*} \; .$$

4 Numerical example

In order to demonstrate the applicability and usefulness of the MRT decomposition, we derive the fund (fund), interest (int), systematic (sys_m), and unsystematic mortality (unsys_m) risk components of a return-of-premium GMDB within a variable annuity (VA). VAs are deferred, unit-linked annuity contracts, and GMDBs are common embedded riders that guarantee a minimal amount paid upon the policyholder's death (see Bauer et al. (2008) for details on VAs with guarantees). With nearly USD 2 trillion in net assets, VAs account for almost a quarter of the US insurance industry's total assets (Insurance Information Institute, 2017).

We assume that the VA is offered against a single premium P_0 paid at time 0, which is fully invested in a fund $S = (S(t))_{0\le t\le T^*}$. If the policyholder dies during the deferment period $[0, T]$, the GMDB guarantees that the death benefit paid at the end of the year of death equals at least the single premium P_0 (return of premium death benefit). We focus on the insurer's risk from the GMDB guarantee and assume that the company charges no fee for this embedded rider. Thus, the policyholder's account value equals $A(t) = P_0 \frac{S(t)}{S(0)},\ t \in [0, T]$, and if identical contracts are issued to m homogeneous individuals, the total discounted loss of the insurance company is:

$$L = \sum_{k=1}^{T} \left(N(t_k) - N(t_{k-1})\right) e^{-\int_0^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\}, \tag{9}$$

where $r(t)$ denotes the time-t interest rate, and $t_k = k,\ k = 0, 1, \ldots, T$. Note that a single upfront fee on top of P_0 would not change $R = L - \mathbb{E}(L)$, thus leading to the same MRT decomposition.

We model the fund process as a geometric Brownian motion with drift μ_S and volatility σ_S:

$$dS(t) = \mu_S S(t)dt + \sigma_S S(t)dW_S(t), \ \ S(0) > 0,$$

where $W_S = (W_S(t))_{0\le t\le T^*}$ denotes a $\mathbb{P}$-Brownian motion. The short rate $r = (r(t))_{0\le t\le T^*}$ is assumed to follow a positive Cox-Ingersoll-Ross (CIR) process:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r \sqrt{r(t)} dW_r(t), \ \ r(0) > 0,$$

where κ, θ, $\sigma_r \in \mathbb{R}$, $2\kappa\theta \geq \sigma_r^2$, and $W_r = (W_r(t))_{0\le t\le T^*}$ is a $\mathbb{P}$-Brownian motion. Moreover, following Dahl and Møller (2006), we assume that under $\mathbb{P}$ the mortality intensity process $\mu = (\mu(t))_{0\le t\le T^*}$ follows a positive time-inhomogeneous CIR process:

$$d\mu(t,a) = (\gamma(t,a) - \delta(t,a)\mu(t,a))dt + \sigma_\mu(t,a)\sqrt{\mu(t,a)} dW_\mu(t), \ \ \mu(0,a) = \mu^0(a),$$

where a denotes the policyholder's age at time 0, $W_\mu = (W_\mu(t))_{0\le t\le T^*}$ is a $\mathbb{P}$-Brownian motion, the initial mortality intensities $\mu^0(a+t) = b_1 + b_2 \times b_3^{a+t}$ are assumed to follow the Gompertz-Makeham mortality law, and:

$$\gamma(t,a) = \frac{1}{2}\hat{\sigma}^2 \mu^0(a+t), \ \ \delta(t,a) = \hat{\delta} - \frac{\frac{\partial}{\partial t}\mu^0(a+t)}{\mu^0(a+t)}, \ \ \sigma_\mu(t,a) = \hat{\sigma}\sqrt{\mu^0(a+t)},$$

for some deterministic parameters $b_1, b_2, b_3, \hat{\delta}$, and $\hat{\sigma}$. We consider a single age cohort, i.e. we fix the initial age a, so that we no longer indicate the dependency on the age cohort but just write $\mu(t)$ and $\sigma_\mu(t)$ etc. Since we assume that $W_S,\ W_r$, and W_μ are independent, one-dimensional Brownian motions, the volatility function of the process $X = (S, r, \mu)^\top$ is

$$\sigma(t,x) = \mathrm{diag}\{\sigma_S x_1, \sigma_r\sqrt{x_2}, \sigma_\mu(t)\sqrt{x_3}\}.$$

Thus, it follows that $\det\{\sigma(t,x)\} \neq 0$ for all $t \in [0, T^*]$ and all values x the process $X(t),\ t \in [0, T^*]$, assumes.

Since X is a Markov process, we can rely on Proposition 4 to determine the MRT decomposition:

$$R = L - \mathbb{E}(L) = R_{\text{fund}} + R_{\text{int}} + R_{\text{sys\_m}} + R_{\text{unsys\_m}}$$

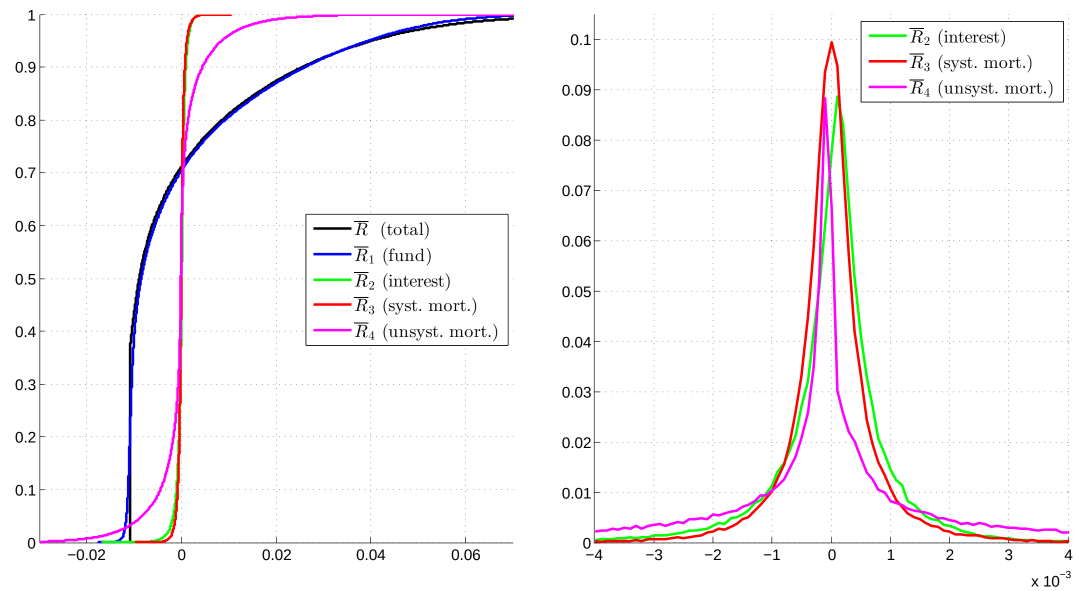
into fund (S), interest (r), systematic mortality (μ), and unsystematic mortality (N) risk components. More precisely, following Equation (4), we can represent the discrete death benefit guarantee payment as a difference of two discrete survival cash flows. The corresponding conditional expectations have to be solved numerically, and we rely on Monte Carlo simulations to obtain the distributions of the risk components. We refer to Part B of the E-Companion for details.

For the numerical calculations, we consider $m = 100$ and $m = 10,000$ GMDB contracts with maturity $T = 15$ and single premium $P_0 = 100{,}000$. All policyholders are assumed to be $a = 50$ years old at time 0. For the mortality model, we adopt the parameter values for year 2003, case II, males, from Tables 1 to 3 in Dahl and Møller (2006): $b_1 = 0.000134,\ b_2 = 0.0000353,\ b_3 = 1.1020,\ \hat{\delta} = 0.008$, and $\hat{\sigma} = 0.02$. For the interest model, we assume $\kappa = 0.2,\ \theta = 0.025,\ \sigma_r = 0.075$, and $r(0) = 0.0029$. The parameters of the geometric Brownian motion are set to $\mu_S = 0.06$ and $\sigma_S = 0.22$.

We focus on the distributions scaled by the number of policyholders in the portfolio and the single premium, i.e. we consider $\overline{R} = \frac{1}{mP_0}R,\ \overline{R}_i = \frac{1}{mP_0}R_i,\ i \in \{\text{fund}, \text{int}, \text{sys\_m}, \text{unsys\_m}\}$. The resulting empirical distribution functions of the total risk $\overline{R}$ and each of the risk components for $m = 100$ are shown in Figure 1(a). We find that the distribution function of the fund risk component is right-skewed while the distribution functions of all other risk components are approximately symmetric. Moreover, the plots indicate that the fund is the most relevant risk driver since the distribution of the risk component closely resembles the distribution of the total risk. This seems intuitive since the fund value determines whether and to what extent the GMDB guarantee is in the money in case of death.

For $m = 100$ contracts, the randomness of the number of deaths within $[0, T]$, which trigger possible payoffs, also seems to be rather high: the range of likely outcomes of the unsystematic mortality risk component is rather wide compared to the ranges of the interest risk component and the systematic mortality risk component. To further illustrate their relationship, we sort the respective outcomes into equally spaced bins of size 0.0001 and plot the corresponding relative frequencies in Figure 1(b). We observe that the tails of the interest risk are heavier than the tails of the systematic mortality risk, but considerably lighter than the tails of the unsystematic mortality risk.

The resulting decomposition can now be used to allocate risk capital as cast by a homogeneous risk measure to the different risk sources via the so-called Euler principle. More precisely, for



(a) Empirical distribution functions (b) Relative frequencies

Figure 1: GMDB portfolio with $m = 100$ contracts – empirical distribution function and relative frequencies using bins of size 0.0001

a homogeneous risk measure ρ and assuming differentiability, by Euler's homogeneous function theorem we can determine the risk capital contribution of the each risk source r_i via:

$$\rho(\overline{R}) = \sum_{i \in \{\text{fund,int,sys\_m,unsys\_m}\}} \underbrace{\left. \frac{\partial \rho\left(a_{\text{fund}}\overline{R}_{\text{fund}} + a_{\text{int}}\overline{R}_{\text{int}} + a_{\text{sys\_m}}\overline{R}_{\text{sys\_m}} + a_{\text{unsys\_m}}\overline{R}_{\text{unsys\_m}}\right)}{\partial a_i} \right|_{a_i=1}}_{=r_i}. \tag{10}$$

Table 2 provides results for three risk measures: standard deviation (Std. Dev.), value-at-risk at the 99.5% level ($\text{VaR}_{0.995}$), and tail-value-at-risk at the 99% level ($\text{TVaR}_{0.99}$) for $m = 100$ and $m = 10,000$. For each risk measure, we report the total risk capital $\rho(\overline{R})$ (per unit per contract) as well as the risk contributions according to the Euler principle (absolute and as a percentage of the sum of the four risk contributions), where we use finite difference approximations for the derivatives in (10). As a result of the numerical approximations, we observe a (slightly) negative contribution of unsystematic mortality risk under VaR and the allocated values do not perfectly add up to the total risk capital $\rho(\overline{R})$ – although the deviation is small (0.0008 for VaR).

The allocated risk contributions confirm our observations from the empirical distribution functions and the relative frequencies. For $m = 100$, the fund risk makes up between about 73% and 89% of the total risk capital, depending on the risk measure, whereas unsystematic mortality risk is the second-most significant component accounting for between roughly 10% and 26%. For

Capital Allocations

ρ	$\rho(\overline{R})$ (total)	r_{fund}	% (fund)	r_{int}	% (interest)	$r_{\text{sys\_m}}$	% (syst. mort.)	$r_{\text{unsys\_m}}$	% (unsyst. mort.)
$m = 100$									
Std. Dev.	0.0179	0.0160	89.3%	0.0001	0.3%	0.0000	0.2%	0.0018	10.1%
$\text{VaR}_{0.995}$	0.0780	0.0619	78.6%	0.0023	2.9%	0.0001	0.1%	0.0145	18.4%
$\text{TVaR}_{0.99}$	0.0813	0.0592	72.9%	0.0008	1.0%	0.0004	0.5%	0.0208	25.6%
$m = 10,000$									
Std. Dev.	0.0169	0.0168	99.2%	0.0001	0.3%	0.0000	0.3%	0.0000	0.2%
$\text{VaR}_{0.995}$	0.0660	0.0656	99.7%	0.0000	0.1%	0.0005	0.7%	-0.0003	-0.5%
$\text{TVaR}_{0.99}$	0.0680	0.0657	96.6%	0.0011	1.7%	0.0008	1.2%	0.0004	0.6%

Table 2: Total risk capital under different risk measures ρ and the corresponding Euler risk contributions in absolute terms and relative to the sum of the four Euler risk contributions for the GMDB portfolio

$m = 10,000$, on the other hand, unsystematic mortality risk diversifies in line with our results from Section 3.4, and fund risk is even more pronounced accounting for between 99.6% (under TVaR) and even 99.7% (under VaR) of total capital. However, an important caveat is that fund risk may be hedged, whereas at least for the systematic mortality risk hedging opportunities are scarce. We leave the further exploration of risk decompositions of hedged positions for future research (for a more detailed study in the context of annuity conversion options based on the methods presented in this paper, see Schilling, 2017).

5 Conclusion

The present paper provides a detailed analysis of risk decomposition approaches. After discussing a list of properties we posit a *meaningful* risk decomposition should satisfy, we propose a novel *MRT decomposition* that satisfies all of these desirable properties – as opposed to other approaches applied in the quantitative risk and insurance literature. We discuss its formal definition, calculation, and properties in a relatively general life insurance setting, and provide several examples for illustration.

Key directions for future research include the generalization of the setting to a broader class of driving processes and to settings beyond life insurance. While some extensions to other insurance applications are underway (see e.g. Jetses (2018) for a generalization to multi-state insurance models and applications in health insurance), we believe our approach will also prove useful for risk decomposition problems in other domains.

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E-Companion to "Dynamic Risks into Risk Components"

This E-Companion collects proofs of all the statements in the main text (Part A) and provides details on the calculations in our examples (Part B).

A Proofs of Statements

Proof of Lemma 1

The first assertion follows from Lando (1998, p. 102), the second assertion follows from Bielecki and Rutkowski (2004, p. 268), and the third assertion follows from Bielecki and Rutkowski (2004, p. 145). □

Proof of Lemma 2

1. Since the drift vector θ is $\mathbb{G}$-adapted with continuous paths, it follows that A_i^W is a predictable finite variation process. Since M_i^W is a local martingale and $X_i(t) = X_i(0) + M_i^W(t) + A_i^W(t)$ for all $t \in [0, T^*]$, A_i^W is a compensator of X_i. The uniqueness follows by Theorem 34 in Protter (2005).

2. By the assumptions, A^N is a predictable finite variation process and M^N is a martingale (for the latter, cf. Bielecki and Rutkowski, 2004, Proposition 5.1.3). Thus, A^N is a compensator of N and the uniqueness again follows by Theorem 34 in Protter (2005). □

Proof of Proposition 1

Applying the martingale representation theorem for point processes in combination with Brownian motions (Björk, 2011, Theorem 4.1.2) to the martingale $M(t) = \mathbb{E}\big(L - \mathbb{E}(L)\big|\mathcal{F}_t^{W,N}\big)$, $0 \leq t \leq T^*$, together with the $\mathcal{F}_{T^*}^{W,N}$-measurability of L, it follows that there exist $\mathbb{F}^{W,N}$-predictable processes $\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_d^W, \psi^N : [0, T^*] \times \Omega \to \mathbb{R}$ such that:

$$R = L - \mathbb{E}(L) = \int_0^{T^*} \widetilde{\psi}^W(t) dW(t) + \int_0^{T^*} \psi^N(t) dM^N(t), \tag{11}$$

where $\widetilde{\psi}^W = (\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_d^W)$. Since $n = d$ and $\det \sigma(t) \neq 0$ by assumption, the inverse of σ, denoted by σ^{-1}, exists (and is unique). Thus, if $\psi_i^W(t) = \sum_{j=1}^d \widetilde{\psi}_j^W(t)\sigma_{ji}^{-1}(t)$, $i = 1, \ldots, n$, denotes the i-th entry of the vector $\widetilde{\psi}^W(t)\sigma^{-1}(t)$, the first summand of (11) can be transformed into:

$$\int_0^{T^*} \widetilde{\psi}^W(t)dW(t) = \int_0^{T^*} \widetilde{\psi}^W(t)\sigma^{-1}(t)\sigma(t)dW(t) = \sum_{i=1}^{n} \int_0^{T^*} \psi_i^W(t)dM_i^W(t),$$

which together with (11) proves the existence of the MRT decomposition (5).

Since $\langle W_i, W_j\rangle(t) = 0$ for all $i \neq j$, and $\langle W_i, M^N\rangle(t) = 0$ for $i = 1, \dots, d$, the Itô isometry yields:

$$\mathbb{E}\left((L - \mathbb{E}(L))^2\right) = \sum_{i=1}^{d} \mathbb{E}\bigg(\Big(\int_0^{T^*} \tilde{\psi}_i^W(t)dW_i(t)\Big)^2\bigg) + \mathbb{E}\bigg(\Big(\int_0^{T^*} \psi^N(t)dM^N(t)\Big)^2\bigg).$$

Thus, by the square integrability of L, all integrals in (11) are square integrable, and in particular (6) holds.

To show uniqueness, suppose there exist $\mathbb{F}^{W,N}$-predictable processes $\widetilde{\xi}_1^W, \dots, \widetilde{\xi}_d^W, \xi^N : [0, T^*] \times \Omega \to \mathbb{R}$ such that:

$$L - \mathbb{E}(L) = \int_0^{T^*} \widetilde{\xi}^W(t)dW(t) + \int_0^{T^*} \xi^N(t)dM^N(t).$$

Then we have $\int_0^{T^*} \big(\widetilde{\psi}^W(t) - \widetilde{\xi}^W(t)\big)dW(t) + \int_0^{T^*} \big(\psi^N(t) - \xi^N(t)\big)\, dM^N(t) = 0$. From Andersen et al. (1997, p. 78), we know that the predictable quadratic variation of $M^N(t)$ equals $\langle M^N, M^N\rangle(t) = \int_0^t (m - N(s-))\mu(s)ds$. Together with the Itô isometry, we thus obtain:

$$0 = \sum_{i=1}^{d} \mathbb{E}\bigg(\int_0^{T^*} \big(\widetilde{\psi}_i^W(t) - \widetilde{\xi}_i^W(t)\big)^2 dt\bigg) + \mathbb{E}\bigg(\int_0^{T^*} \big(\psi^N(t) - \xi^N(t)\big)^2 (m - N(t-))\mu(t)dt\bigg).$$

Since μ is assumed to be positive, it directly follows that $\tilde{\psi}_i^W = \tilde{\xi}_i^W$ $\lambda \otimes \mathbb{P}$-almost surely, $i = 1, \dots, d$, and that $\psi^N = \xi^N$ on $\{(t, \omega) \in [0, T^*] \times \Omega : N(t-) < m\}$ with respect to $\lambda \otimes \mathbb{P}$. Finally, the uniqueness of $\psi_1^W, \dots, \psi_n^W, \psi^N$ is a result of the uniqueness of $\widetilde{\psi}_1^W, \dots, \widetilde{\psi}_n^W, \psi^N$ and the uniqueness of the inverse of σ. □

Proof of Proposition 2

Obviously, the risk components $R_1, \dots, R_{n+1}$ are random variables, and $R = \sum_{i=1}^{n+1} R_i$, so that P1 and P6* (and thus also P6) are satisfied. The uniqueness property P3 directly follows from Proposition 1 and the fact that $\Delta M^N(t) = 0$ on $\{(t, \omega) \in [0, T^*] \times \Omega : N(t-) = m\}$.

To simplify the proof of the remaining properties, let $\psi_i = \psi_i^W,\ M_i = M_i^W,\ i = 1, \dots, n$, and $\psi_{n+1} = \psi^N,\ M_{n+1} = M^N$. Furthermore, we write $Z = (Z_1, \dots, Z_{n+1}) = (X_1, \dots, X_n, N)$.

Assume that $(R, Z_1, \dots, Z_{n+1}) \overset{\text{MRT}}{\leftrightarrow} (R_1, \dots, R_{n+1})$.

P2: Let $i \in \{1, \dots, n+1\}$. Assume that R is $\sigma(Z_i)$-measurable and that Z_i is independent of $Z_{i-} = (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_{n+1})$. This directly implies that R is independent of Z_{i-}. Furthermore, since $\det \sigma(t) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, we have $\mathcal{F}_t^{W,N} = \mathcal{F}_t^Z = \mathcal{F}_t^{Z_i} \vee \mathcal{F}_t^{Z_{i-}}$, where $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{0 \le t \le T^*}$, $\mathbb{F}^{Z_i} = (\mathcal{F}_t^{Z_i})_{0 \le t \le T^*}$, and $\mathbb{F}^{Z_{i-}} = (\mathcal{F}_t^{Z_{i-}})_{0 \le t \le T^*}$ denote the augmented filtrations generated by $Z,\ Z_i$, and Z_{i-}, respectively. Thus:

$$R(t) = \sum_{j=1}^{n+1} \int_0^t \psi_j(s) dM_j(s) = \mathbb{E}\left(R \middle| \mathcal{F}_t^{W,N}\right) = \mathbb{E}\left(R \middle| \mathcal{F}_t^{Z_i} \vee \mathcal{F}_t^{Z_{i-}}\right) = \mathbb{E}\left(R \middle| \mathcal{F}_t^{Z_i}\right).$$

This implies that the process $(R(t))_{0 \le t \le T^*}$ is independent of each process $Z_j,\ j \neq i$, so that the predictable covariation process satisfies $\langle R, Z_j \rangle(t) = 0$ for all $j \neq i,\ 0 \le t \le T^*$.

(a) Assume that $i = n+1$. Then $\langle M_i, Z_j \rangle(t) = \langle M^N, X_j \rangle(t) = 0$ for all $j \neq i$, so that:

$$\begin{aligned} 0 = d\langle R, Z_j \rangle(t) &= \sum_{k=1}^{n+1} \psi_k(t) d\langle M_k, Z_j \rangle(t) = \sum_{k=1}^{n} \psi_k(t) d\langle M_k, Z_j \rangle(t) \\ &= \sum_{k=1}^{n} \psi_k(t) \sigma_{k,\cdot}(t) \sigma_{j,\cdot}^\top(t) dt, \quad j \neq i,\ 0 \le t \le T^*, \end{aligned} \tag{12}$$

where $\sigma_{k,\cdot}(t)$ denotes the k-th row of $\sigma(t)$. For any $0 \le t \le T^*$, this yields the linear system of equations $A_t^\top \psi_t = 0$, where $\psi_t = (\psi_1(t), \dots, \psi_n(t))^\top$ and $A_t = \sigma(t)\sigma(t)^\top$, so that $\det A_t^\top = (\det \sigma(t))^2 \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, implying $\psi_t = 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Thus, we have $R_j = \int_0^{T^*} \psi_j(t) dM_j(t) = 0$ almost surely for all $j \neq i$.

(b) Now assume that $i \neq n+1$ (w.l.o.g. $i = 1$). Then we know that:

$$\begin{aligned} 0 = d\langle R, Z_{n+1} \rangle(t) &= \sum_{k=1}^{n+1} \psi_k(t) d\langle M_k, Z_{n+1} \rangle(t) = \psi_{n+1}(t) d\langle M_{n+1}, Z_{n+1} \rangle(t) \\ &= \psi_{n+1}(t) d\langle M_{n+1}, M_{n+1} \rangle(t), \end{aligned}$$

so by the Itô isometry it follows that:

$$\mathbb{E}\left(\left[\int_0^{T^*} \psi_{n+1}(t) dM_{n+1}(t)\right]^2\right) = \mathbb{E}\left(\int_0^{T^*} \psi_{n+1}^2(t) d\langle M_{n+1}, M_{n+1} \rangle(t)\right) = 0,$$

and thus $R_{n+1} = \int_0^{T^*} \psi_{n+1}(t) dM_{n+1}(t) = 0$ almost surely.

Since Z_1 is by assumption independent of Z_{1-} and thus independent of Z_j for all $j = 2, \ldots, n+1$, it follows that $\sigma_{1,\cdot}(t)\sigma_{j,\cdot}(t)^\top dt = d\langle Z_1, Z_j\rangle(t) = 0$ for all $j \notin \{1, n+1\}$. Thus, for $A_t = \sigma(t)\sigma(t)^\top$, we obtain:

$$A_t = \begin{pmatrix} \sigma_{1,\cdot}(t)\sigma_{1,\cdot}(t)^\top & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \tilde{A}_t & \\ 0 & & & \end{pmatrix}, \quad \tilde{A}_t = \begin{pmatrix} \sigma_{2,\cdot}(t)\sigma_{2,\cdot}(t)^\top & \ldots & \sigma_{2,\cdot}(t)\sigma_{n,\cdot}(t)^\top \\ \vdots & & \vdots \\ \sigma_{n,\cdot}(t)\sigma_{2,\cdot}(t)^\top & \ldots & \sigma_{n,\cdot}(t)\sigma_{n,\cdot}(t)^\top \end{pmatrix},$$

and since $0 \neq \det A_t = \sigma_{1,\cdot}(t)\sigma_{1,\cdot}(t)^\top \det \tilde{A}_t$, $\det \tilde{A}_t \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Furthermore, following the same calculation steps as in (12) for $j \notin \{1, n+1\}$ and using $\langle M_1, Z_j\rangle(t) = d\langle Z_1, Z_j\rangle(t) = 0$ and $\langle M_{n+1}, Z_j\rangle(t) = 0,\ j \notin \{1, n+1\}$, we obtain the linear system $\tilde{A}_t^\top \tilde{\psi}_t = 0$, where $\tilde{\psi}_t = (\psi_2(t), \ldots, \psi_n(t))^\top$. Since $\det \tilde{A}_t \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, it follows that $\tilde{\psi}_t = 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, and thus $R_j = \int_0^{T^*} \psi_j(t) dM_j(t) = 0$ almost surely for all $j \notin \{1, n+1\}$.

P4: Consider a permutation $\pi : \{1, \ldots, n+1\} \to \{1, \ldots, n+1\}$. Let $(R, Z_{\pi(1)}, \ldots, Z_{\pi(n+1)}) \overset{\text{MRT}}{\leftrightarrow} (\tilde{R}_1, \ldots, \tilde{R}_{n+1})$ with $\tilde{R}_i = \int_0^{T^*} \tilde{\psi}_i(t) dM_{\pi(i)}(t)$ for $i = 1, \ldots, n+1$, where $\tilde{\psi}_i$ are $\mathbb{F}$-predictable processes. Since:

$$\sum_{i=1}^{n+1} \int_0^{T^*} \tilde{\psi}_i(t) dM_{\pi(i)}(t) = \sum_{i=1}^{n+1} \tilde{R}_i \overset{\text{P6}^*}{=} R \overset{\text{P6}^*}{=} \sum_{i=1}^{n+1} R_i = \sum_{i=1}^{n+1} \int_0^{T^*} \psi_i(t) dM_i(t)$$
$$= \sum_{i=1}^{n+1} \int_0^{T^*} \psi_{\pi(i)}(t) dM_{\pi(i)}(t),$$

P4 follows by the uniqueness of the MRT decomposition.

P5: Let $\tilde{Z}_i(t) = f_i(Z_i(t)),\ i = 1, \ldots, n+1$, where the functions $f_i : \mathbb{R} \to \mathbb{R}$ are smooth and invertible, and consider $(R, \tilde{Z}_1, \ldots, \tilde{Z}_{n+1}) \overset{\text{MRT}}{\leftrightarrow} (\tilde{R}_1, \ldots, \tilde{R}_{n+1})$.

For each $i \neq n+1$, we have by Itô's lemma:

$$d\tilde{Z}_i(t) = f_i'(X_i(t)) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t) + \left(f_i'(X_i(t))\theta(t) + \frac{1}{2} f_i''(X_i(t)) \sum_{j=1}^{d} \sigma_{ij}^2(t) \right) dt.$$

Thus, $(\tilde{Z}_1, \ldots, \tilde{Z}_n)$ is again an Itô process as in Assumption 1 and by Lemma 2 the corresponding compensated risk processes equal:

$$\tilde{M}_i(t) = f_i'(X_i(t)) dM_i(t), \ \ i = 1, \ldots, n.$$

As a result, for $i = 1, \dots, n$, the MRT risk components equal:

$$\tilde{R}_i = \int_0^{T^*} \tilde{\psi}_i(t) d\tilde{M}_i(t) = \int_0^{T^*} \tilde{\psi}_i(t) f_i'(X_i(t)) dM_i(t). \tag{13}$$

For $i = n + 1$, we have:

$$\begin{aligned}
\tilde{Z}_{n+1}(t) &= f_{n+1}(N(0)) + \sum_{0<s\leq t} \left(f_{n+1}(N(s)) - f_{n+1}(N(s-))\right) \\
&= f_{n+1}(N(0)) + \sum_{0<s\leq t} \underbrace{\left[\sum_{k=0}^{m} \mathbf{1}_{\{N(s-)=k\}} \left(f_{n+1}(k+1) - f_{n+1}(k)\right)\right]}_{=a(s)} (N(s) - N(s-)) \\
&= f_{n+1}(N(0)) + \int_0^t a(s) dN(s) \\
&= f_{n+1}(N(0)) + \int_0^t a(s) dM_{n+1}(s) + \int_0^t a(s)(m - N(s-))\mu(s) ds,
\end{aligned}$$

exploiting in the second equality that $\mathbb{P}(\tau_x^i = \tau_x^j) = 0$ for $i \neq j$ (Bielecki and Rutkowski, 2004, p. 269). Since $a(s) \neq 0$ (invertible) and predictable, $\tilde{A}_{n+1}(t) = \int_0^t a(s)(m - N(s-))\mu(s) ds$ is a predictable finite variation process and $\tilde{M}_{n+1}(t) = \int_0^t a(s) dM_{n+1}(s)$ is a local martingale. Thus:

$$\tilde{R}_{n+1} = \int_0^{T^*} \tilde{\psi}_{n+1}(t) d\tilde{M}_{n+1}(t) = \int_0^{T^*} \tilde{\psi}_{n+1}(t) a(t) dM_{n+1}(t). \tag{14}$$

The uniqueness of the MRT decomposition together with (13) and (14) implies that $R_i = \tilde{R}_i$ almost surely, $i = 1, \dots, n+1$, and thus P5. □

Remark 5 *While the MRT decomposition in* (5) *is formally defined in terms of the Itô process* X *and the counting process* N, *in the proof of P5 we consider a generalized notion in terms of an Itô process and the jump process* $\int_0^\cdot a(s) dN(s)$. *However, since the generalization is straightforward and to keep the presentation in Section 3 concise, we accept this slight inconsistency.*

Proof of Proposition 3

The proof relies on three lemmas that, while building on previous literature, present new results on martingale representation of life insurance payment processes (see Remark 6 below for details).

Lemma 3 ***(Discrete survival cash flow)*** *Let* Z *be a random variable of the form* $Z = (m - N(T))\, F$, $T \in [0, T^*]$, *where* F *is* $\mathcal{G}_{T^*}$*-measurable and integrable. Then there exist* $\mathbb{G}$*-predictable processes*

$\varphi_1, \ldots, \varphi_d$ *such that:*

$$\mathbb{E}\left(e^{-\Gamma(T)}F\middle|\, \mathcal{G}_t\right) = \mathbb{E}\left(e^{-\Gamma(T)}F\right) + \sum_{i=1}^{d} \int_0^t \varphi_i(u)dW_i(u), \;\; t \leq T^*, \tag{15}$$

and the martingale representation of Z *is given by:*

$$\begin{aligned} Z = \; & \mathbb{E}(Z) + \sum_{i=1}^{d} \int_0^{T^*} \left[(m - N(t-))\, e^{\Gamma(t)}\boldsymbol{1}_{[0,T]}(t) + (m - N(T))\, e^{\Gamma(T)}\boldsymbol{1}_{(T,T^*]}(t)\right] \varphi_i(t)dW_i(t) \\ & - \int_0^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(T)}F\middle|\mathcal{G}_t\right) dM^N(t). \end{aligned} \tag{16}$$

Proof of Lemma 3

Since $U = (U(t))_{0\leq t\leq T^*}$ with $U(t) = \mathbb{E}\left(e^{-\Gamma(T)}F\middle|\, \mathcal{G}_t\right)$ is a $\mathbb{G}$-martingale, it follows by the martingale representation theorem that there exist predictable processes $\varphi_1, \ldots, \varphi_d$ such that (15) holds.

We first show the lemma for a single policyholder with remaining lifetime τ_x^i, i.e. $m = 1$ and $\mathbb{F} = \mathbb{G} \vee \mathbb{I}^i$ for some arbitrary but fixed $i \in \{1, \ldots, m\}$. Define $L_i(t) = \mathbf{1}_{\{\tau_x^i > t\}} e^{\Gamma(t)}$ and $\tilde{L}_i(t) = \mathbb{E}\left(L_i(T)\middle|\, \mathcal{F}_t\right)$. Since $L_i(t)$ is an $\mathbb{F}$-martingale (Bielecki and Rutkowski, 2004, p. 152), it follows that $\tilde{L}_i(t) = L_i(t)$ for $t \leq T$ and $\tilde{L}_i(t) = L_i(T)$ for $t \geq T$. Furthermore, $U(T^*) = e^{-\Gamma(T)}F$, which implies $Z_i = \mathbf{1}_{\{\tau_x^i > T\}}F = \tilde{L}_i(T^*)U(T^*)$. Thus, applying the Itô integration by parts formula (Protter, 2005, p. 68) to the product $\tilde{L}_i(t)U(t)$ and considering the continuity of $U(t)$ yields:

$$\begin{aligned} Z_i &= \tilde{L}_i(0)U(0) + \int_0^{T^*} \tilde{L}_i(t-)dU(t) + \int_0^{T^*} U(t)d\tilde{L}_i(t) + [\tilde{L}_i, U](T^*) \\ &= L_i(0)U(0) + \int_0^{T^*} \left[L_i(t-)\mathbf{1}_{[0,T]}(t) + L_i(T)\mathbf{1}_{(T,T^*]}(t)\right] dU(t) + \int_0^{T} U(t)dL_i(t) + [L_i, U](T), \end{aligned} \tag{17}$$

where the second equality follows from the definition of $\tilde{L}_i$. Using $\mathbf{1}_{\{\tau_x^i > 0\}} = 1$ a.s. (which follows from the assumptions on μ), 2 in Lemma 1, and the $\mathcal{G}_{T^*}$-measurability of F, we have that:

$$L_i(0)U(0) \overset{\text{a.s.}}{=} \mathbb{E}\left(e^{-\Gamma(T)}F\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{\tau_x^i > T\}}\middle|\, \mathcal{G}_{T^*}\right) F\right) = \mathbb{E}\left(\mathbf{1}_{\{\tau_x^i > T\}}F\right) = \mathbb{E}(Z_i).$$

Also note that:

$$M_i^N(t) = \mathbf{1}_{\{\tau_x^i \leq t\}} - \int_0^t \mathbf{1}_{\{\tau_x^i > s-\}}\mu(s)ds = \mathbf{1}_{\{\tau_x^i \leq t\}} - \int_0^{t\wedge\tau_x^i} \mu(s)ds.$$

Thus, since the $\mathbb{G}$-adapted cumulative mortality intensity Γ of τ_x^i is continuous and increasing, Proposition 5.1.3 i) in Bielecki and Rutkowski (2004) implies that:

$$dL_i(t) = -L_i(t-)dM_i^N(t).$$

Plugging in the definitions of L_i and M_i^N, this can be further written as:

$$dL_i(t) = -e^{\Gamma(t)}\left(\mathbf{1}_{\{\tau_x^i>t-\}}d\mathbf{1}_{\{\tau_x^i\le t\}} - \mathbf{1}_{\{\tau_x^i>t-\}}\mathbf{1}_{\{\tau_x^i>t\}}\mu(t)dt\right) = -e^{\Gamma(t)}dM_i^N(t).$$

Moreover, $[L_i, U](t) = 0$ for every $t \in [0, T^*]$ (Bielecki and Rutkowski, 2004, p. 160). Thus, using the martingale representation of $U(t)$, equation (17) becomes:

$$Z_i = \mathbb{E}(Z_i) + \sum_{j=1}^{d}\int_0^{T^*}\left[L_i(t-)\mathbf{1}_{[0,T]}(t) + L_i(T)\mathbf{1}_{(T,T^*]}(t)\right]\varphi_j(t)dW_j(t) - \int_0^T U(t)e^{\Gamma(t)}dM_i^N(t).$$

Together with the continuity and adaptedness of μ, this proves the statement of the proposition for any single policyholder.

In the portfolio case, where $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^m \mathbb{I}^i$, the conditional independence of the τ_x^i's implies that $\mathbb{E}\left(Z_i|\,\mathcal{F}_t\right) = \mathbb{E}\left(Z_i|\,\mathcal{G}_t \vee \mathcal{I}_t^i\right)$. Thus, by using the conditionally identical distribution of $\tau_x^i,\ i = 1,\dots,m$, the proposition follows for the entire portfolio from applying the previous part of the proof to each summand of $Z = \sum_{i=1}^m \mathbf{1}_{\{\tau_x^i>T\}}F$ separately and adding the respective decompositions. □

Lemma 4 ***(Continuous survival cash flow)*** *Let Z be a random variable of the form $Z = \int_0^T (m - N(v))F(v)dv,\ T \in [0, T^*]$, where $F = (F(t))_{0\le t\le T}$ is a $\mathbb{G}$-predictable process with $\mathbb{E}\left(\sup_{t\in[0,T]}|F(t)|\right) < \infty$. Then there exist $\mathbb{G}$-predictable processes $\varphi_1,\dots,\varphi_d$ such that:*

$$\mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\middle|\,\mathcal{G}_t\right) = \mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\right) + \sum_{i=1}^d\int_0^t \varphi_i(u)dW_i(u),\ \ t \le T, \tag{18}$$

and the martingale representation of Z is given by:

$$Z = \mathbb{E}(Z) + \sum_{i=1}^d\int_0^T (m - N(t-))e^{\Gamma(t)}\varphi_i(t)dW_i(t) - \int_0^T\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)}F(v)\middle|\mathcal{G}_t\right)dv\,dM^N(t). \tag{19}$$

In particular, if additionally $\sup_{t\in[0,T]}\mathbb{E}\left([F(t)]^2\right) < \infty$, then:

$$\varphi_i(t) = \int_t^T \varphi_i^v(t)dv,\ \ t \le T, \tag{20}$$

where $\varphi_i^v,\ i=1,\dots,d,\ v\in[0,T]$, *are the* $\mathbb{G}$*-predictable integrands of the martingale representation of* $e^{-\Gamma(v)}F(v)$ *(cf.* (15)*).*

Proof of Lemma 4

To begin with, note that by the martingale representation theorem, there exist predictable processes $\varphi_1,\dots,\varphi_d$ such that (18) holds. Again, we first show the statement for a single policyholder with remaining lifetime τ_x^i, i.e. $m=1$ and $\mathbb{F}=\mathbb{G}\vee\mathbb{I}^i$ for an arbitrary but fixed $i\in\{1,\dots,m\}$. Since F is assumed to be $\mathbb{G}$-predictable with $\mathbb{E}\left(\sup_{t\in[0,T]}|F(t)|\right)<\infty$, it follows from Proposition 5.1.2 in Bielecki and Rutkowski (2004) that:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\tau_x^i>v\}}F(v)dv\,\middle|\,\mathcal{F}_t\right)\\
&=\int_0^t \mathbf{1}_{\{\tau_x^i>v\}}F(v)dv+L_i(t)\,\mathbb{E}\left(\int_t^T e^{-\Gamma(v)}F(v)dv\,\middle|\,\mathcal{G}_t\right)\\
&=\int_0^t \mathbf{1}_{\{\tau_x^i>v\}}F(v)dv-L_i(t)\int_0^t e^{-\Gamma(v)}F(v)dv+L_i(t)\,\mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\,\middle|\,\mathcal{G}_t\right),
\end{aligned}\tag{21}$$

where $L_i(t)=\mathbf{1}_{\{\tau_x^i>t\}}e^{\Gamma(t)}$. Note that Proposition 5.1.2 in Bielecki and Rutkowski (2004) actually requires $\int_0^T F(v)dv$ to be bounded. However, via dominated convergence it can be shown that the result still holds if F satisfies $\mathbb{E}\left(\sup_{t\in[0,T]}|F(t)|\right)<\infty$ (Biagini et al., 2016, p. 22, already point out a possible relaxation to $\mathbb{E}\left(\sup_{t\in[0,T]}|F(t)|^2\right)<\infty$). As in the proof of Lemma 3, it follows by applying integration by parts that:

$$L_i(t)\int_0^t e^{-\Gamma(v)}F(v)dv=\int_0^t \mathbf{1}_{\{\tau_x^i>s-\}}F(s)ds-\int_0^t\left(\int_0^s e^{-\Gamma(v)}F(v)dv\right)e^{\Gamma(s)}dM_i^N(s)$$

and:

$$\begin{aligned}
L_i(t)\,\mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\,\middle|\,\mathcal{G}_t\right)&=\mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\right)+\sum_{i=1}^d\int_0^t \mathbf{1}_{\{\tau_x^i>s-\}}e^{\Gamma(s)}\varphi_i(s)dW_i(s)\\
&\quad-\int_0^t \mathbb{E}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\,\middle|\,\mathcal{G}_s\right)e^{\Gamma(s)}dM_i^N(s),
\end{aligned}$$

where $M_i^N(t) = \mathbf{1}_{\{\tau_x^i \le t\}} - \int_0^t \mathbf{1}_{\{\tau_x^i > s-\}} \mu(s) ds$. Summing up the representations of all summands from (21) and using the $\mathcal{F}_T$-measurability of $\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv$, we obtain:

$$\begin{aligned}\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv = \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) dv\right) + \sum_{i=1}^d \int_0^T \mathbf{1}_{\{\tau_x^i > t-\}} e^{\Gamma(t)} \varphi_i(t) dW_i(t) \\ - \int_0^T \mathbb{E}\left(\int_t^T e^{\Gamma(t)-\Gamma(v)} F(v) dv \,\middle|\, \mathcal{G}_t\right) dM_i^N(t).\end{aligned} \tag{22}$$

Since we assume that $\mathbb{E}\left(\sup_{t\in[0,T]} |F(t)|\right) < \infty$, the theorem of Fubini-Tonelli together with the construction of τ_x^i implies that:

$$\mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) dv\right) = \mathbb{E}\left(\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv\right),$$

and the theorem of Fubini-Tonelli for conditional expectations yields:

$$\int_0^T \mathbb{E}\left(\int_t^T e^{\Gamma(t)-\Gamma(v)} F(v) dv \,\middle|\, \mathcal{G}_t\right) dM_i^N(t) = \int_0^T \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)} F(v) \,\middle|\, \mathcal{G}_t\right) dv\, dM_i^N(t),$$

so that (19) follows from (22).

Next we prove (20). By the martingale representation theorem, there exist for every $v \in [0, T]$ $\mathbb{G}$-predictable processes $\varphi_1^v, \ldots, \varphi_d^v$ such that:

$$\mathbb{E}\left(e^{-\Gamma(v)} F(v) \,\middle|\, \mathcal{G}_t\right) = \mathbb{E}\left(e^{-\Gamma(v)} F(v)\right) + \sum_{i=1}^d \int_0^t \varphi_i^v(u) \mathbf{1}_{[0,v]}(u) dW_i(u), \quad t \in [0, T].$$

Thus, using the theorem of Fubini-Tonelli and the stochastic Fubini theorem (Protter, 2005, Theorem 65), it follows that:

$$\begin{aligned}\mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) dv \,\middle|\, \mathcal{G}_t\right) &= \int_0^T \mathbb{E}\left(e^{-\Gamma(v)} F(v) \,\middle|\, \mathcal{G}_t\right) dv \\ &= \int_0^T \mathbb{E}\left(e^{-\Gamma(v)} F(v)\right) dv + \sum_{i=1}^d \int_0^T \int_0^t \varphi_i^v(u) \mathbf{1}_{[0,v]}(u) dW_i(u)\, dv \\ &= \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) dv\right) + \sum_{i=1}^d \int_0^t \int_0^T \varphi_i^v(u) \mathbf{1}_{[0,v]}(u) dv\, dW_i(u),\end{aligned}$$

where for applying the stochastic Fubini theorem it is sufficient to note that:

$$\mathbb{E}\left(\int_0^t \int_0^T \left[\varphi_i^v(u)\right]^2 \mathbf{1}_{[0,v]}(u) dv\, du\right) \le \int_0^T \mathbb{E}\left(\int_0^T \left[\varphi_i^v(u)\right]^2 \mathbf{1}_{[0,v]}(u) du\right) dv$$
$$\le T \sup_{v\in[0,T]} \mathbb{E}\left(\int_0^T \left[\varphi_i^v(u)\right]^2 \mathbf{1}_{[0,v]}(u) du\right) \le T \sup_{v\in[0,T]} \mathbb{E}\left(\left[e^{-\Gamma(v)} F(v)\right]^2\right)$$
$$\le T \sup_{v\in[0,T]} \mathbb{E}\left([F(v)]^2\right) < \infty.$$

The uniqueness of the martingale representation finally implies (20).

By the conditional independence assumption on $\tau_x^i,\ i = 1,\dots,m,$ we have in the portfolio case with $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^m \mathbb{I}^i$ that $\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv \middle| \mathcal{F}_t\right) = \mathbb{E}\left(\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv \middle| \mathcal{G}_t \vee \mathcal{I}_t^i\right)$. Thus, the statement for the portfolio directly follows by applying the obtained equation to each summand $\int_0^T \mathbf{1}_{\{\tau_x^i > v\}} F(v) dv,\ i = 1,\dots,m,$ separately and adding the respective decompositions. □

Lemma 5 ***(Continuous cash flow contingent on death)*** *Let Z be a random variable of the form $Z = \int_0^T F(v) dN(v),\ T \in [0,T^*]$, where $F = (F(t))_{0\le t\le T}$ is a continuous and $\mathbb{G}$-predictable process with $\mathbb{E}\left(\sup_{t\in[0,T]} |F(t)|\right) < \infty$. Then there exist $\mathbb{G}$-predictable processes $\varphi_1,\dots,\varphi_d$ such that for $t \le T$:*

$$\mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v) \middle| \mathcal{G}_t\right) = \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v)\right) + \sum_{i=1}^d \int_0^t \varphi_i(u) dW_i(u), \tag{23}$$

and the martingale representation of Z is given by:

$$\begin{aligned} Z = \mathbb{E}(Z) &+ \sum_{i=1}^d \int_0^T (m - N(t-))\, e^{\Gamma(t)} \varphi_i(t) dW_i(t) \\ &- \int_0^T \left[\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)} F(v)\mu(v) \middle| \mathcal{G}_t\right) dv - F(t)\right] dM^N(t). \end{aligned} \tag{24}$$

In particular, if additionally $\sup_{t\in[0,T]} \mathbb{E}\left([F(t)]^4\right) < \infty$ and $\sup_{t\in[0,T]} \mathbb{E}\left(\mu^4(t)\right) < \infty$, then:

$$\varphi_i(t) = \int_t^T \varphi_i^v(t) dv,\ \ t \le T, \tag{25}$$

where $\varphi_i^v,\ i = 1,\dots,d,\ v \in [0,T]$, are the $\mathbb{G}$-predictable integrands of the martingale representation of $e^{-\Gamma(v)} F(v)\mu(v)$ (cf. (15)).

Proof of Lemma 5

To begin with, note that by the martingale representation theorem, there exist predictable processes $\varphi_1, \ldots, \varphi_d$ such that (23) holds. Since F is continuous, it follows from the definition of Lebesgue integrals that:

$$Z = \int_0^T F(v) dN(v) = \sum_{i=1}^m \mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i). \tag{26}$$

Again, we first show the statement for a single policyholder with remaining lifetime τ_x^i, i.e. $m = 1$ and $\mathbb{F} = \mathbb{G} \vee \mathbb{I}^i$ for an arbitrary but fixed $i \in \{1, \ldots, m\}$. Note that $\mathbf{1}_{\{\tau_x^i \leq t\}} F(\tau_x^i)$ is $\mathcal{F}_t$-measurable, so that:

$$\mathbb{E}\left(\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{F}_t\right) = \mathbb{E}\left(\mathbf{1}_{\{t < \tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{F}_t\right) + \mathbf{1}_{\{\tau_x^i \leq t\}} F(\tau_x^i). \tag{27}$$

Since F is assumed to be $\mathbb{G}$-predictable with $\mathbb{E}\left(\sup_{t \in [0,T]} |F(t)|\right) < \infty$, it follows from Corollary 5.1.3 in Bielecki and Rutkowski (2004) that:

$$\begin{aligned}
\mathbb{E}\left(\mathbf{1}_{\{t < \tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{F}_t\right) &= \mathbf{1}_{\{\tau_x^i > t\}} \mathbb{E}\left(\int_t^T e^{\Gamma(t) - \Gamma(v)} F(v) d\Gamma(v) \middle| \mathcal{G}_t\right) \\
&= L_i(t) \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v) \middle| \mathcal{G}_t\right) - L_i(t) \int_0^t e^{-\Gamma(v)} F(v) d\Gamma(v),
\end{aligned}$$

where $L_i(t) = \mathbf{1}_{\{\tau_x^i > t\}} e^{\Gamma(t)}$. Again, Proposition 5.1.1 and thus Corollary 5.1.3 in Bielecki and Rutkowski (2004) actually require F to be bounded, but a generalization to non-bounded F satisfying $\mathbb{E}\left(\sup_{t \in [0,T]} |F(t)|\right) < \infty$ can be shown via dominated convergence (Biagini et al., 2016, p. 19, already point out a possible relaxation to $\mathbb{E}\left(\sup_{t \in [0,T]} |F(t)|^2\right) < \infty$). As in the proof of Lemma 3, it then follows by applying integration by parts to both addends that:

$$\begin{aligned}
&\mathbb{E}\left(\mathbf{1}_{\{t < \tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{F}_t\right) \\
&= \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v)\right) + \sum_{i=1}^d \int_0^t \mathbf{1}_{\{\tau_x^i > s-\}} e^{\Gamma(s)} \varphi_i(s) dW_i(s) \\
&\quad - \int_0^t \mathbb{E}\left(\int_s^T e^{\Gamma(s) - \Gamma(v)} F(v) d\Gamma(v) \middle| \mathcal{G}_s\right) dM_i^N(s) - \int_0^t \mathbf{1}_{\{\tau_x^i > s\}} F(s) d\Gamma(s),
\end{aligned}$$

where $M_i^N(t) = \mathbf{1}_{\{\tau_x^i \leq t\}} - \int_0^t \mathbf{1}_{\{\tau_x^i > s-\}} \mu(s) ds$. On the other hand, we obtain by (26) that:

$$\mathbf{1}_{\{\tau_x^i \leq t\}} F(\tau_x^i) = \int_0^t F(s) d\mathbf{1}_{\{\tau_x^i \leq s\}} = \int_0^t F(s) dM_i^N(s) + \int_0^t F(s) \mathbf{1}_{\{\tau_x^i > s\}} d\Gamma(s).$$

Summing up the representations of the two summands in (27) and using the $\mathcal{F}_T$-measurability of $\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i)$, we obtain:

$$
\begin{aligned}
\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i) = \mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v) \right) + \sum_{i=1}^{d} \int_0^T \mathbf{1}_{\{\tau_x^i > t-\}} e^{\Gamma(t)} \varphi_i(t) dW_i(t) \\
- \int_0^T \left[\mathbb{E}\left(\int_t^T e^{\Gamma(t)-\Gamma(v)} F(v) d\Gamma(v) \middle| \mathcal{G}_t \right) - F(t) \right] dM_i^N(t).
\end{aligned} \tag{28}
$$

Corollary 5.1.3 in Bielecki and Rutkowski (2004) implies that:

$$
\mathbb{E}\left(\int_0^T e^{-\Gamma(v)} F(v) d\Gamma(v) \right) = \mathbb{E}\left(\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i) \right),
$$

and since $\mathbb{E}\left(\int_t^T |F(v)| \, e^{\Gamma(t)-\Gamma(v)} \mu(v) dv \right) \leq \mathbb{E}\left(\sup_{v \in [0,T]} |F(v)| \right) < \infty$, the theorem of Fubini-Tonelli for conditional expectations yields:

$$
\mathbb{E}\left(\int_t^T F(v) e^{\Gamma(t)-\Gamma(v)} d\Gamma(v) \middle| \mathcal{G}_t \right) = \int_t^T \mathbb{E}\left(F(v) e^{\Gamma(t)-\Gamma(v)} \mu(v) \middle| \mathcal{G}_t \right) dv,
$$

so that (24) follows from (28). The proof of (25) works analogously to the proof of (20), additionally using the Cauchy-Schwarz inequality.

By the conditional independence assumption on $\tau_x^i, \ i = 1, \ldots, m$, we have in the portfolio case with $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^m \mathbb{I}^i$ that $\mathbb{E}\left(\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{F}_t \right) = \mathbb{E}\left(\mathbf{1}_{\{\tau_x^i \leq T\}} F(\tau_x^i) \middle| \mathcal{G}_t \vee \mathcal{I}_t^i \right)$. Thus, the statement for the portfolio directly follows by applying the obtained equation to each summand $\mathbf{1}_{\{\tau_x^i \leq t\}} F(\tau_x^i), \ i = 1, \ldots, m$, separately and adding the respective decompositions. □

Remark 6 *As mentioned in Remark 4, the above lemmas are closely related to (quadratic) hedging of insurance liabilities, and we heavily rely on this line of research. Specifically, for a single policyholder, the proof of Lemma 3 mainly follows the ideas of the proof of Proposition 5.2.2 in Bielecki and Rutkowski (2004), albeit we modify their result so that it fits our later application and extend it to an entire (homogeneous) portfolio. For F $\mathcal{G}_T$-measurable instead of (more generally) $\mathcal{G}_{T^*}$-measurable, similar results (usually in a specific process setting) have been derived in the context of quadratic hedging strategies, see e.g. Barbarin (2008, Prop. 4.10, Prop. 5.11), Biagini et al. (2016, Prop. 3.5), Biagini et al. (2013, Prop. 2, Prop. 9), and Biagini and Schreiber (2013, Lemma 4.2). Most of them also consider entire portfolios. For Lemma 4, except for some details, the proof of the first part* (19) *mainly follows the proof of Proposition 4.12 in Barbarin (2008). The specification* (20) *may simplify the derivation of* (18). *For bounded F, it has already been shown in Biagini et al. (2013, Proposition 5). For Lemma 5, the proof of the first part* (24) *relies on a generalization of Proposition 4.11 in Barbarin (2008) and Proposition 4 in Biagini et al. (2013).*

Similar results were independently derived in Section 3.3 of Biagini et al. (2016) and in Section 4 of Biagini and Schreiber (2013). We added the specification (25) *in analogy to* (20).

Returning to the main proof, the integrands in Proposition 3 follow directly from Lemma 3, Lemma 4, and Lemma 5 together with the Clark-Ocone formula (Di Nunno et al., 2009, p. 196) and the proof of Proposition 1. Since each L is square integrable as a result of the respective assumptions, the uniqueness follows from Proposition 2. □

Proof of Proposition 4

Since $n = d$, $\det \sigma(t, X(t)) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, and each L is square integrable as a result of the respective assumptions, the uniqueness of the decompositions follows by Proposition 2. Furthermore, the assumptions imply that X is a Markov process, which together with the factorization lemma yields for all cases i) to iv) below that:

$$\mathbb{E}\left(\cdot \middle| \mathcal{G}_t\right) = \mathbb{E}\left(\cdot \middle| X(t)\right) \tag{29}$$

is a function of $X(t)$. Define $G(t) = \int_0^t g(s, X(s))ds,\ 0 \leq t \leq T$, and note that (Shreve, 2004, p. 480):

$$d[G,G](t) = d[G,\Gamma](t) = d[\Gamma,\Gamma](t) = d[G,X_i](t) = d[\Gamma,X_i](t) = 0. \tag{30}$$

1. The assumption on the form of $C_{0,T}$ together with (29) yields that:

$$\begin{aligned}\mathbb{E}\left(C_{0,T}\middle| \mathcal{G}_t\right) &= e^{-G(t)}\mathbb{E}\left(e^{-\int_t^T g(s,X(s))ds} h(X(T))\middle| \mathcal{G}_t\right) = e^{-G(t)} f(t, X(t)) \\ &= \tilde{f}(t, G(t), X(t)).\end{aligned}$$

   Since f is assumed to be smooth, this holds for $\tilde{f}$ as well. Thus, Itô's lemma yields for $0 \leq t \leq T$ (Protter, 2005, Theorem 33):

$$\mathbb{E}\left(C_{0,T}\middle| \mathcal{G}_t\right) - \mathbb{E}\left(C_{0,T}\right) = \sum_{i=1}^n \int_0^t e^{-G(s)} \frac{\partial f}{\partial x_i}(s, X(s)) dM_i^W(s) + \int_0^t a(s)ds,$$

   where $a = (a(t))_{0\leq t\leq T^*}$ is short-hand for all ds-quantities. We have used (30) and that $(t, G(t), X(t))$ has continuous paths. The right-hand side $\mathbb{E}\left(C_{0,T}\middle| \mathcal{G}_t\right) - \mathbb{E}\left(C_{0,T}\right)$ is a martingale. On the other hand, the stochastic integrals with respect to $M_i^W,\ i = 1, \ldots, n$, are martingales as well. Thus, it follows by the uniqueness of the Doob-Meyer decomposition (Protter, 2005, Theorem 16) that the ds-term vanishes. Since $C_{0,T}$ is $\mathcal{G}_T$-measurable, the statement follows.

2. We consider the two cases $T > t_k$ and $T \leq t_k$ separately. Together they yield the result. In both cases, $T > t_k$ and $T \leq t_k$, we derive the MRT decomposition with the help of Lemma 3. For this, we determine the martingale representation of $e^{-\Gamma(t_k)} C_{a,k}$ less its expectation.

   (a) If $T > t_k$, we consider the decomposition:

$$\begin{aligned} &e^{-\Gamma(t_k)} C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)} C_{a,k}\right) \\ &= \left[e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right) - \mathbb{E}\left(e^{-\Gamma(t_k)} C_{a,k}\right)\right] + e^{-\Gamma(t_k)} \left[C_{a,k} - \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right)\right], \end{aligned} \tag{31}$$

   and derive the martingale representations of the two parts separately. The assumption on the form of $C_{a,k}$ together with (29) yield for $0 \leq t \leq t_k$ that:

$$\begin{aligned} \mathbb{E}\left(e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right) \middle| \mathcal{G}_t\right) &= e^{-\Gamma(t)} e^{-G(t)} \mathbb{E}\left(e^{\Gamma(t) - \Gamma(t_k)} e^{G(t) - G(T)} h(X(T)) \middle| \mathcal{G}_t\right) \\ &= e^{-\Gamma(t)} e^{-G(t)} f^A(t, X(t)) \\ &= \tilde{f}^A(t, \Gamma(t), G(t), X(t)). \end{aligned}$$

   Since f^A is assumed to be smooth, this holds for $\tilde{f}^A$ as well. Thus, Itô's formula yields for $0 \leq t \leq t_k$ (Protter, 2005, Theorem 33):

$$\begin{aligned} &\mathbb{E}\left(e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right) \middle| \mathcal{G}_t\right) - \mathbb{E}\left(e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right)\right) \\ &= \sum_{i=1}^{n} \int_0^t e^{-\Gamma(s)} e^{-G(s)} \frac{\partial f^A}{\partial x_i}(s, X(s)) dM_i^W(s) + \int_0^t a(s) ds, \end{aligned}$$

   where $a = (a(t))_{0 \leq t \leq T^*}$ is short-hand for all ds-quantities. We have used (30) and that $(t, \Gamma(t), G(t), X(t))$ has continuous paths. By the same arguments as in i) the ds-term vanishes, and since $e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right)$ is $\mathcal{G}_{t_k}$-measurable, it follows that:

$$e^{-\Gamma(t_k)} \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right) - \mathbb{E}\left(e^{-\Gamma(t_k)} C_{a,k}\right) = \sum_{i=1}^{n} \int_0^{t_k} e^{-\Gamma(s)} e^{-G(s)} \frac{\partial f^A}{\partial x_i}(s, X(s)) dM_i^W(s).$$

   Furthermore, applying part i) to $C_{a,k}$ it holds that:

$$e^{-\Gamma(t_k)} \left[C_{a,k} - \mathbb{E}\left(C_{a,k} \middle| \mathcal{G}_{t_k}\right)\right] = \sum_{i=1}^{n} \int_{t_k}^{T} e^{-\Gamma(t_k)} e^{-G(s)} \frac{\partial f^B}{\partial x_i}(s, X(s)) dM_i^W(s).$$

In total, using (31) we have:

$$\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\\
&= \sum_{i=1}^{n}\int_0^T \Big[e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))\mathbf{1}_{[0,t_k]}(s)\\
&\qquad\qquad + e^{-\Gamma(t_k)}e^{-G(s)}\frac{\partial f^B}{\partial x_i}(s,X(s))\mathbf{1}_{(t_k,T]}(s)\Big]dM_i^W(s).
\end{aligned}$$

This implies equation (15) using the equality:

$$\begin{aligned}
&\sum_{i=1}^{n}\int_0^t \tilde{\varphi}_i(u)dM_i^W(u) = \int_0^t \tilde{\varphi}(u)dM^W(u) = \int_0^t \tilde{\varphi}(u)\sigma(u)dW(u)\\
&= \sum_{j=1}^{d}\int_0^t (\tilde{\varphi}(u)\sigma(u))_j dW_j(u),
\end{aligned} \tag{32}$$

where $\tilde{\varphi} = (\tilde{\varphi}_1,\ldots,\tilde{\varphi}_n)$ is any vector, $M^W = (M_1^W,\ldots,M_n^W)$, and $(\cdot)_j$ denotes the j-th component of a vector. The statement then follows by Lemma 3.

(b) If $T \le t_k$, we consider the decomposition:

$$\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\\
&= \left[\mathbb{E}\left(e^{-\Gamma(t_k)}\middle|\,\mathcal{G}_T\right)C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\right] + \left[e^{-\Gamma(t_k)} - \mathbb{E}\left(e^{-\Gamma(t_k)}\middle|\,\mathcal{G}_T\right)\right]C_{a,k}
\end{aligned}$$

and again derive the martingale representations of the two parts separately. Analogously to above, we obtain:

$$\mathbb{E}\left(e^{-\Gamma(t_k)}\middle|\,\mathcal{G}_T\right)C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)}C_{a,k}\right) = \sum_{i=1}^{n}\int_0^T e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))dM_i^W(s),$$

and:

$$\left[e^{-\Gamma(t_k)} - \mathbb{E}\left(e^{-\Gamma(t_k)}\middle|\,\mathcal{G}_T\right)\right]C_{a,k} = \sum_{i=1}^{n}\int_T^{t_k} e^{-\Gamma(s)}C_{a,k}\frac{\partial f^C}{\partial x_i}(s,X(s))dM_i^W(s),$$

so that:

$$
\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathbb{E}\left(e^{-\Gamma(t_k)}C_{a,k}\right) \\
&= \sum_{i=1}^{n}\int_0^{t_k}\Big[e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))\mathbf{1}_{[0,T]}(s) \\
&\qquad\qquad + e^{-\Gamma(s)}C_{a,k}\frac{\partial f^C}{\partial x_i}(s,X(s))\mathbf{1}_{(T,t_k]}(s)\Big]dM_i^W(s).
\end{aligned}
$$

This implies equation (15) using (32). The statement then follows by Lemma 3.

3. The assumption on the form of $C_a(v)$ together with (29) yield that, for each $v \in [0,T]$:

$$
\begin{aligned}
\mathbb{E}\left(e^{-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right) &= e^{-\Gamma(t)}e^{-G(t)}\mathbb{E}\left(e^{-\int_t^v[\mu(s,X(s))+g(s,X(s))]ds}h(X(v))\middle|\,\mathcal{G}_t\right) \\
&= e^{-\Gamma(t)}e^{-G(t)}f^v(t,X(t)) \\
&= \tilde{f}^v(t,\Gamma(t),G(t),X(t)),\ \ t \le v,
\end{aligned}
$$

where $f^v : [0,v]\times\mathbb{R}^n \to \mathbb{R}$. Since f^v is assumed to be smooth, this holds for $\tilde{f}^v$ as well. Thus, Itô's formula yields for $t \le v$ (Protter, 2005, Theorem 33):

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right) - \mathbb{E}\left(e^{-\Gamma(v)}C_a(v)\right) \\
&= \sum_{i=1}^{n}\int_0^t e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^v}{\partial x_i}(s,X(s))dM_i^W(s) + \int_0^t a(s)ds,
\end{aligned}
$$

where $a = (a(t))_{0\le t\le T^*}$ is short-hand for all ds-quantities. We have used (30) and that $(t,\Gamma(t),G(t),X(t))$ has continuous paths. By the same arguments as in i), the ds-term has to vanish. Thus, exploiting (32) we obtain by Lemma 4 for $t \in [0,T]$ (for $t > T$ all integrands are zero) that:

$$
\begin{aligned}
\psi_i^W(t) &= (m - N(t-))e^{\Gamma(t)}\int_t^T \varphi_i^v(t)dv \\
&= (m - N(t-))e^{-G(t)}\int_t^T \frac{\partial f^v}{\partial x_i}(t,X(t))dv,\ \ i = 1,\dots,n,
\end{aligned}
$$

and:

$$\begin{aligned}\psi^N(t) &= -\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right)dv \\ &= -\int_t^T e^{-G(t)}\mathbb{E}\left(e^{-\int_t^v[\mu(s,X(s))+g(s,X(s))]ds}h(X(v))\middle|\,\mathcal{G}_t\right)dv \\ &= -e^{-\int_0^t g(s,X(s))ds}\int_t^T f^v(t,X(t))dv.\end{aligned}$$

4. As in part iii), the assumption on the form of $C_{ad}(t)$ and $\mu(t)$ together with (29) yield that, for each $v \in [0,T]$:

$$\begin{aligned}&\mathbb{E}\left(e^{-\Gamma(v)}C_{ad}(v)\mu(v)\middle|\,\mathcal{G}_t\right) \\ &= e^{-\Gamma(t)}e^{-G(t)}\mathbb{E}\left(e^{-\int_t^v[\mu(s,X(s))+g(s,X(s))]ds}h(X(v))\mu(v,X(v))\middle|\,\mathcal{G}_t\right) \\ &= e^{-\Gamma(t)}e^{-G(t)}f^v(t,X(t)) \\ &= \tilde{f}^v(t,\Gamma(t),G(t),X(t)), \quad t \le v,\end{aligned}$$

where $f^v : [0,v] \times \mathbb{R}^n \to \mathbb{R}$. Thus, the integrands $\psi_i^W(t),\ i = 1,\dots,n$, of part iv) follow analogously to part iii) using Lemma 5 instead of Lemma 4. Lemma 5 also yields for $t \le T$ (otherwise it is equal to zero) that:

$$\begin{aligned}\psi^N(t) &= -\left[\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(v)}C_{ad}(v)\mu(v)\middle|\,\mathcal{G}_t\right)dv - C_{ad}(t)\right] \\ &= -\left[\int_t^T e^{-G(t)}\mathbb{E}\left(e^{-\int_t^v[\mu(s,X(s))+g(s,X(s))]ds}h(X(v))\mu(v)\middle|\,\mathcal{G}_t\right)dv - C_{ad}(t)\right] \\ &= -\left[e^{-\int_0^t g(s,X(s))ds}\int_t^T f^v(t,X(t))dv - C_{ad}(t)\right]. \ \square\end{aligned}$$

Proof of Proposition 5

The following lemma will simplify the proof.

Lemma 6 *Let* $T \in [0,T^*]$ *be fixed. If* $\sup_{t\in[0,T]}\mathbb{E}\left(\mu^2(t)\right) < \infty$ *and if* $\left(\psi^N(t)\right)_{0\le t\le T}$ *is a* $\mathbb{G}$-*predictable process with* $\sup_{t\in[0,T]}\mathbb{E}\left(\left[\psi^N(t)\right]^4\right) < \infty$, *then:*

$$\frac{1}{m}\int_0^T \psi^N(t)dM^N(t) \xrightarrow[m\to\infty]{L^2} 0.$$

Proof of Lemma 6

We need to show that:

$$\mathbb{E}\left(\left[\frac{1}{m}\int_0^T \psi^N(t)dM^N(t) - 0\right]^2\right) = \frac{1}{m^2}\mathbb{E}\left(\left[\int_0^T \psi^N(t)dM^N(t)\right]^2\right) \xrightarrow[m\to\infty]{} 0.$$

From Andersen et al. (1997, p. 78), we know that the predictable quadratic variation of $M^N(t)$ equals $\langle M^N, M^N\rangle(t) = \int_0^t (m-N(s-))\mu(s)ds$. Since $M^N(t)$ is a martingale, since ψ^N is assumed to be predictable, and since $\mathbb{E}\left(\int_0^T \left[\psi^N(t)\right]^2 d\langle M^N, M^N\rangle(t)\right) < \infty$ by the calculations below, it follows that $\int_0^T \psi^N(t)dM^N(t)$ is a square integrable martingale and that the Itô isometry applies (for both, see Klebaner, 2005, p. 234) yielding:

$$\begin{aligned} \frac{1}{m^2}\mathbb{E}\left(\left[\int_0^T \psi^N(t)dM^N(t)\right]^2\right) &= \frac{1}{m^2}\mathbb{E}\left(\int_0^T \left[\psi^N(t)\right]^2 \underbrace{(m - N(t-))}_{\leq m}\mu(t)dt\right) \\ &\leq \frac{1}{m}\mathbb{E}\left(\int_0^T \left[\psi^N(t)\right]^2 \mu(t)dt\right). \end{aligned} \tag{33}$$

Since by assumption $C_1 = \sup_{t\in[0,T]} \mathbb{E}\left(\left[\psi^N(t)\right]^4\right) < \infty$ and $C_2 = \sup_{t\in[0,T]} \mathbb{E}\left(\mu^2(t)\right) < \infty$, the theorem of Fubini-Tonelli and the Cauchy-Schwarz inequality yield:

$$\begin{aligned} \mathbb{E}\left(\int_0^T \left[\psi^N(t)\right]^2 \mu(t)dt\right) &= \int_0^T \mathbb{E}\left(\left[\psi^N(t)\right]^2 \mu(t)\right) dt \\ &\overset{\text{Cauchy-Schwarz}}{\leq} \int_0^T \sqrt{\mathbb{E}\left(\left[\psi^N(t)\right]^4\right)\mathbb{E}\left(\mu^2(t)\right)}dt \leq \int_0^T \sqrt{C_1C_2} = T\sqrt{C_1C_2} = C < \infty. \end{aligned}$$

Together with (33), we obtain:

$$0 \leq \frac{1}{m^2}\mathbb{E}\left(\left[\int_0^T \psi^N(t)dM^N(t)\right]^2\right) \leq \frac{1}{m}\mathbb{E}\left(\int_0^T \left[\psi^N(t)\right]^2 \mu(t)dt\right) \leq \frac{1}{m}C \xrightarrow[m\to\infty]{} 0. \ \square$$

Returning to the proof of Proposition 5, note that any conditional expectation $\mathbb{E}(\cdot|\mathcal{G}_t)$ is predictable, since it is by definition $\mathcal{G}_t$-measurable and $\mathcal{G}_t$ is left-continuous as a result of the continuity of Brownian motions.

1. The process $(\psi^N_{ak}(t))_{0\leq t\leq t_k}$ defined by $\psi^N_{ak}(t) = \mathbb{E}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\,\mathcal{G}_t\right)$ for all $t \in [0, t_k]$ is predictable. Furthermore, applying Jensen's inequality for conditional expectations (Protter,

2005, p. 11), and using that $\Gamma(t)$ is non-decreasing in t, it follows that:

$$\begin{aligned}
&\sup_{t\in[0,t_k]} \mathbb{E}\left(\left[\psi_{ak}^N(t)\right]^4\right) = \sup_{t\in[0,t_k]} \mathbb{E}\left(\left[\mathbb{E}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\,\mathcal{G}_t\right)\right]^4\right)\\
&\le \sup_{t\in[0,t_k]} \mathbb{E}\left(\mathbb{E}\left(\left[e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\right]^4\middle|\,\mathcal{G}_t\right)\right) \le \sup_{t\in[0,t_k]} \mathbb{E}\left(\mathbb{E}\left(\left[C_{a,k}\right]^4\middle|\,\mathcal{G}_t\right)\right)\\
&= \sup_{t\in[0,t_k]} \mathbb{E}\left(\left[C_{a,k}\right]^4\right) = \mathbb{E}\left(\left[C_{a,k}\right]^4\right) < \infty \quad \text{(by assumption).}
\end{aligned}$$

Since we also assume that $\sup_{t\in[0,t_k]}\mathbb{E}\left(\mu^2(t)\right) < \infty$, the statement follows by Lemma 6.

2. The process $(\psi_a^N(t))_{0\le t\le T}$ defined by $\psi_a^N(t) = \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)ds$ for all $t\in[0,T]$ is predictable. Furthermore, since $0 \le e^{\Gamma(t)-\Gamma(s)} \le 1$ for $s \ge t$ and since $C = \sup_{t\in[0,T]}\mathbb{E}\left(|C_a(t)|\right) < \infty$ as a result of the boundedness of $C_a(t)$, it follows by applying Jensen's inequality for integrals and for conditional expectations (for the latter, cf. Protter, 2005, p. 11) that for any $t\in[0,T]$:

$$\begin{aligned}
\left|\psi_a^N(t)\right| &= \left|\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)ds\right| \le \int_t^T \left|\mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)\right|ds\\
&\le \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}\left|C_a(s)\right|\middle|\mathcal{G}_t\right)ds \le C\,T.
\end{aligned}$$

Thus, we have:

$$\sup_{t\in[0,T]}\mathbb{E}\left(\left[\psi_a^N(t)\right]^4\right) \le \sup_{t\in[0,T]}\mathbb{E}\left([CT]^4\right) = C^4T^4 < \infty.$$

Since we also assume that $\sup_{t\in[0,T]}\mathbb{E}\left(\mu^2(t)\right) < \infty$, the statement follows by Lemma 6.

3. Since $X_m, Y_m, X, Y \in L^2(\mathbb{P})$ and $X_m \xrightarrow{L^2} X,\ Y_m \xrightarrow{L^2} Y$ implies that $X_m + Y_m \xrightarrow{L^2} X + Y$, it is sufficient to show that:

a) $\displaystyle \frac{1}{m}\int_0^T\left[-\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_{ad}(s)\mu(s)\middle|\,\mathcal{G}_t\right)ds\right]dM^N(t) \xrightarrow[m\to\infty]{L^2} 0,$ and

b) $\displaystyle \frac{1}{m}\int_0^T C_{ad}(t)dM^N(t) \xrightarrow[m\to\infty]{L^2} 0.$

Define $\psi_{ad,1}^N(t) = -\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_{ad}(s)\mu(s)ds\middle|\,\mathcal{G}_t\right)$ and $\psi_{ad,2}^N(t) = C_{ad}(t)$ for all $t\in[0,T]$. Note that since by assumption $\sup_{t\in[0,T]}\mathbb{E}\left(\mu^4(t)\right) < \infty$, it also follows by Jensen's inequality that:

$$\sup_{t\in[0,T]}\mathbb{E}\left(\mu^2(t)\right) \le \sup_{t\in[0,T]}\sqrt{\mathbb{E}\left(\mu^4(t)\right)} < \infty.$$

ad a): Since the process $(\psi^N_{ad,1}(t))_{0\leq t\leq T}$ is predictable, since $0 \leq e^{\Gamma(t)-\Gamma(s)} \leq 1$ for $s \geq t$, and since $C_1 = \sup_{t\in[0,T]} \mathbb{E}\left(|C_{ad}(t)|\right) < \infty$ as a result of the boundedness of C_{ad}, it follows by applying Jensen's inequality for integrals and for conditional expectations (for the latter, cf. Protter, 2005, p. 11) that:

$$\begin{aligned}\left|\psi^N_{ad,1}(t)\right| &= \left|\int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}C_{ad}(s)\mu(s)\middle|\,\mathcal{G}_t\right)ds\right|\\ &\leq \int_t^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(s)}\left|C_{ad}(s)\right|\mu(s)\middle|\,\mathcal{G}_t\right)ds\\ &\leq C_1\int_t^T \mathbb{E}\left(\mu(s)|\,\mathcal{G}_t\right)ds \leq C_1\int_0^T \mathbb{E}\left(\mu(s)|\,\mathcal{G}_t\right)ds.\end{aligned}$$

Since by assumption $C_2 = \sup_{t\in[0,T]} \mathbb{E}\left(\mu^4(t)\right) < \infty$, this implies:

$$\begin{aligned}&\sup_{t\in[0,T]} \mathbb{E}\left(\left[\psi^N_{ad,1}(t)\right]^4\right) \leq \sup_{t\in[0,T]} \mathbb{E}\left(\left[C_1\int_0^T \mathbb{E}\left(\mu(s)|\,\mathcal{G}_t\right)ds\right]^4\right)\\ &\overset{(*)}{\leq} \sup_{t\in[0,T]} C_1^4\,\mathbb{E}\left(\int_0^T \mathbb{E}\left(\mu^4(s)\middle|\,\mathcal{G}_t\right)ds\right)\\ &\overset{(**)}{=} \sup_{t\in[0,T]} C_1^4 \int_0^T \mathbb{E}\left(\mu^4(s)\right)ds \leq C_1^4\,C_2\,T < \infty,\end{aligned}$$

where $(*)$ again follows by Jensen's inequality for integrals and conditional expectations and $(**)$ from the theorem of Fubini-Tonelli. Since $\sup_{t\in[0,T]} \mathbb{E}\left(\mu^2(t)\right) < \infty$ as shown above, the statement follows by Lemma 6.

ad b): The process $(\psi^N_{ad,2}(t))_{0\leq t\leq T}$ is predictable. As a result of the boundedness of $C_{ad}(t)$, it also holds $C_1 = \sup_{t\in[0,T]} \mathbb{E}\left(|C_{ad}(t)|\right) < \infty$, so that:

$$\sup_{t\in[0,T]} \mathbb{E}\left(\left[\psi^N_{ad,2}(t)\right]^4\right) = \sup_{t\in[0,T]} \mathbb{E}\left(\left[C_{ad}(t)\right]^4\right) \leq C_1^4 < \infty.$$

Since $\sup_{t\in[0,T]} \mathbb{E}\left(\mu^2(t)\right) < \infty$ as shown above, the statement directly follows by Lemma 6. □

Remark 7 *Convergence in probability of the unsystematic risk components (instead of L^2-convergence) can be shown under less restrictive assumptions, e.g. by applying the (stochastic) dominated convergence theorem similarly as below for the systematic risk components.*

Proof of Proposition 6

The following lemma will simplify the proof.

Lemma 7 *If $\zeta = (\zeta(t))_{0 \leq t \leq T}$ is $\mathbb{G}$-predictable and $\int_0^T \zeta(t)^2 dt < \infty$ almost surely, then for $0 \leq t_k \leq T \leq T^*$:*

$$\frac{1}{m} \int_0^T \left[(m - N(t-))e^{\Gamma(t)} \boldsymbol{1}_{[0,t_k]} + (m - N(t_k))e^{\Gamma(t_k)} \boldsymbol{1}_{(t_k,T]}\right] \zeta(t) dW(t) \xrightarrow[m\to\infty]{P} \int_0^T \zeta(t) dW(t),$$

where $(W(t))_{0 \leq t \leq T^*}$ is a one-dimensional Brownian motion.

Proof of Lemma 7

Define:

$$\zeta_m(t) = \left[\frac{(m - N(t-))}{m} e^{\Gamma(t)} \mathbf{1}_{[0,t_k]}(t) + \frac{(m - N(t_k))}{m} e^{\Gamma(t_k)} \mathbf{1}_{(t_k,T]}(t)\right] \zeta(t).$$

If $\zeta_m = (\zeta_m(t))_{0 \leq t \leq T}$, $m \in \mathbb{N}$, are predictable processes with $\zeta_m(t) \xrightarrow[m\to\infty]{a.s.} \zeta(t)$ for all $t \in [0, T]$, and if there exists a W-integrable process $\alpha = (\alpha(t))_{0 \leq t \leq T}$ such that $|\zeta_m(t)| \leq \alpha(t)$ for all $m \in \mathbb{N}$, $t \in [0, T]$, then the statement of the lemma follows by the dominated convergence theorem for stochastic integrals (Protter, 2005, p. 176) . Since ζ and μ are by assumption predictable, it follows that ζ_m is predictable for each $m \in \mathbb{N}$. Furthermore, since the remaining lifetimes τ_x^i, $i \in \mathbb{N}$, are assumed to be conditionally i.i.d., a conditional version of Kolmogorov's strong law of large numbers (Majerek et al., 2005, p. 154) together with the continuity of $\mu(t)$ yields that:

$$\frac{m - N(t-)}{m} \xrightarrow[m\to\infty]{a.s.} e^{-\int_0^t \mu(s)ds}.$$

As a result, $\zeta_m(t) \xrightarrow[m\to\infty]{a.s.} \zeta(t)$ for all $t \in [0, T]$. Furthermore, since $\frac{m-N(t-)}{m} \leq 1$ and $\mu(t)$ is positive for all $t \in [0, T]$, we have:

$$|\zeta_m(t)| \leq \left[e^{\Gamma(t)} \mathbf{1}_{[0,t_k]}(t) + e^{\Gamma(t_k)} \mathbf{1}_{(t_k,T]}(t)\right] |\zeta(t)| \leq e^{\Gamma(T)} |\zeta(t)| = \alpha(t).$$

Since $\mu(t)$ has continuous paths (particularly on $[0, T]$), it follows that $e^{2\Gamma(T)} < \infty$ a.s. Together with the assumption $\int_0^T \zeta(t)^2 dt < \infty$ a.s., we obtain that $\int_0^T \alpha(t)^2 dt < \infty$ with probability one. Since α is also $\mathbb{G}$-predictable, this implies that α is W-integrable (Klebaner, 2005, p. 96), and the statement follows. $\square$

Returning to the proof of Proposition 6, since $M_i^W(t) = \sum_{k=1}^d \int_0^t \sigma_{ik}(s) dW_k(s),\ 0 \leq t \leq T^*$, it follows that:

$$R_{i,\cdot}^{(m)} = \sum_{k=1}^{d}\sum_{j=1}^{d}\int_0^T \left[(m - N(t-))e^{\Gamma(t)}\mathbf{1}_{[0,t_k]}(t) + (m - N(t_k))e^{\Gamma(t_k)}\mathbf{1}_{(t_k,T]}(t)\right] \times \varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\sigma_{ik}(t)dW_k(t). \tag{34}$$

Because of the additivity of integration and the continuous mapping theorem, it is sufficient to prove the convergence of each summand in (34), $i = 1,\ldots,n,\ j,k = 1,\ldots,d$, separately. For this, by Lemma 7, we only need to show that each $\varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\sigma_{ik}(t)$ is $\mathbb{G}$-predictable with $\int_0^T(\varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\sigma_{ik}(t))^2dt < \infty$ almost surely. We have:

- By assumption, $\sigma(t)$ is $\mathbb{G}$-adapted with continuous paths.
- When determining the inverse of $\sigma(t)$ with Cramer's rule and the necessary determinants with Laplace's formula, it can be seen that $\sigma_{ij}^{-1}(t)$ is a continuous function of the matrix components $\sigma_{ij}(t),\ i = 1,\ldots n,\ j = 1\ldots,d$. So $\sigma_{ij}^{-1}(t)$ has itself continuous paths and is $\mathbb{G}$-adapted.
- In all parts ii), iii), and iv), $\varphi_{j,\cdot}(t)$ is a conditional expectation of the form $\mathbb{E}\left(\cdot|\,\mathcal{G}_t\right)$ or can be transformed into such an expectation using the theorem of Fubini-Tonelli for conditional expecations. As a result, $\varphi_{j,\cdot}(t)$ is by definition $\mathbb{G}$-adapted.
- The $\mathbb{D}_{1,2}$-assumptions in Proposition 3 and particularly the implicit square integrability of the respective quantities yield that:

$$\mathrm{E}\left(\int_0^T \varphi_{j,\cdot}(t)^2dt\right) = \mathrm{E}\left(\left(\int_0^T \varphi_{j,\cdot}(t)dW_j(t)\right)^2\right) < \infty,$$

  implying that $\int_0^T \varphi_{j,\cdot}(t)^2dt < \infty$ almost surely.

Since $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T}$ is left-continuous as a result of the continuity of Brownian motions, every $\mathbb{G}$-adapted process is also $\mathbb{G}$-predictable. Thus, the product $\varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\sigma_{ik}(t)$ is not only $\mathbb{G}$-adapted, but also $\mathbb{G}$-predictable. Furthermore, since $\sigma_{ji}^{-1}(t)\sigma_{ik}(t)$ has continuous paths and $\int_0^T \varphi_{j,\cdot}(t)^2dt < \infty$ almost surely, it follows similarly as in the proof of Lemma 7 that $\int_0^T(\varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)\sigma_{ik}(t))^2dt < \infty$ almost surely. The statement then directly follows by Lemma 7. □

Proof of Corollary 1

1. Since each $L_\cdot^{(m)}$ can be written as the sum of m random variables which are conditionally identically distributed and conditionally independent given the σ-algebra $\mathcal{G}_{T^*}$, the statement

follows by a conditional version of Kolmogorov's strong law of large numbers (Majerek et al., 2005, Theorem 4.2).

2. Since $\mathbb{E}(L_{\cdot}^{(m)}) = m\,\mathbb{E}(L_{\cdot}^{(1)})$ as a result of the conditionally identical distribution of $\tau_x^i,\ i = 1,\ldots,m,$ it follows by part i) that:

$$\frac{1}{m}\left(L_{\cdot}^{(m)} - \mathbb{E}(L_{\cdot}^{(m)})\right) \xrightarrow[m\to\infty]{P} \mathbb{E}(L_{\cdot}^{(1)}|\mathcal{G}_{T^*}) - \mathbb{E}(L_{\cdot}^{(1)}).$$

Furthermore, by Proposition 5 (note that L^2-convergence implies convergence in probability) and Proposition 6 we also have:

$$\frac{1}{m}\left(L_{\cdot}^{(m)} - \mathbb{E}(L_{\cdot}^{(m)})\right) = \sum_{i=1}^{n+1} \frac{1}{m} R_{i,\cdot}^{(m)} \xrightarrow[m\to\infty]{P} \sum_{i=1}^{n} \int_0^T \sum_{j=1}^{d} \varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)dM_i^W(t).$$

Since the limit in probability is almost surely unique, it follows that:

$$\mathbb{E}(L_{\cdot}^{(1)}|\mathcal{G}_{T^*}) - \mathbb{E}(L_{\cdot}^{(1)}) = \sum_{i=1}^{n} \int_0^T \sum_{j=1}^{d} \varphi_{j,\cdot}(t)\sigma_{ji}^{-1}(t)dM_i^W(t),$$

which is an MRT decomposition. By the uniqueness of the MRT decomposition (Proposition 2), each risk component R_i^* thus equals almost surely the limit in probability of $\frac{1}{m}R_{i,\cdot}^{(m)},\ i = 1,\ldots,n+1,$ so that the statement follows. $\square$

B Calculation of the MRT Decomposition Examples

Details on the Examples from Section 2.4

The MRT decompositions for Examples 1, 2, and 3 follow by calculating the expected values as detailed in Proposition 4.1 (where $h \equiv 0$ in all three examples) and taking derivatives.

Similarly, by Proposition 4.1, the MRT components for Example 4 are also given by the partial derivatives of the conditional expectation of the risk R. We have:

$$\begin{aligned}
& \mathrm{E}[L|Z_1(t), Z_2(t)] \\
= \;& \mathrm{E}\left[(Z_1(t) + \sigma_1\,(W_1(1) - W_1(t)) \times \max\left\{K - Z_2(t) - \sigma_2\,(W_2(1) - W_2(t)), 0\right\}\right| Z_1(t), Z_2(t)] \\
= \;& Z_1(t) \times \mathrm{E}\left[(K - Z_2(t))\,\mathbf{1}_{\{(K - Z_2(t))/\sigma_2 > W_2(1) - W_2(t)\}}|Z_2(t)\right] \\
& \quad -\sigma_2\,Z_1(t) \times \mathrm{E}\left[(W_2(1) - W_2(t))\,\mathbf{1}_{\{(K - Z_2(t))/\sigma_2 > W_2(1) - W_2(t)\}}|Z_2(t)\right] \\
= \;& Z_1(t)\,(K - Z_2(t))\,\Phi\left(\frac{K - Z_2(t)}{\sigma_2\sqrt{1-t}}\right) - \sigma_2\,Z_1(t)\mathrm{E}\left[(W_2(1) - W_2(t))\,\mathbf{1}_{\{(K - Z_2(t))/\sigma_2 > W_2(1) - W_2(t)\}}\right],
\end{aligned}$$

where Φ is the standard normal cumulative distribution function. Note that for a normal random variable $X \sim N(0, s^2)$, we obtain:

$$\mathrm{E}[X\,\mathbf{1}_{\{X<y\}}] = \int_{-\infty}^{y} x\,\frac{1}{\sqrt{2\pi}s}\,e^{-x^2/_{2s^2}}\,dx = -\frac{1}{\sqrt{2\pi}}\,s\,e^{-y^2/_{2s^2}}.$$

Thus:

$$\mathrm{E}[L|Z_1(t),Z_2(t)] = Z_1(t)\,(K-Z_2(t))\,\Phi\left(\frac{K-Z_2(t)}{\sigma_2\sqrt{1-t}}\right) + \sigma_2\,Z_1(t)\,\frac{\sqrt{1-t}}{\sqrt{2\pi}}\,\exp\left\{-\frac{(K-Z_2(t))^2}{2\,\sigma_2^2\,(1-t)}\right\}.$$

Taking derivatives, we obtain:

$$\frac{\partial}{\partial Z_1(t)}\,\mathrm{E}[L|Z_1(t),Z_2(t)] \;=\; (K-Z_2(t))\,\Phi\left(\frac{K-Z_2(t)}{\sigma_2\sqrt{1-t}}\right) + \sigma_2\,\frac{\sqrt{1-t}}{\sqrt{2\pi}}\,\exp\left\{-\frac{(K-Z_2(t))^2}{2\,\sigma_2^2\,(1-t)}\right\}$$

and:

$$\begin{aligned}\frac{\partial}{\partial Z_2(t)}\,\mathrm{E}[L|Z_1(t),Z_2(t)] &= -Z_1(t)\,\Phi\left(\frac{K-Z_2(t)}{\sigma_2\sqrt{1-t}}\right) - Z_1(t)\,(K-Z_2(t))\,\frac{1}{\sigma_2\sqrt{1-t}}\,\phi\left(\frac{K-Z_2(t)}{\sigma_2\sqrt{1-t}}\right)\\ &\quad + Z_1(t)\,(K-Z_2(t))\,\frac{1}{\sqrt{1-t}\,\sigma_2\sqrt{2\pi}}\,\exp\left\{-\frac{(K-Z_2(t))^2}{2\,\sigma_2^2\,(1-t)}\right\}\\ &= -Z_1(t)\,\Phi\left(\frac{K-Z_2(t)}{\sigma_2\sqrt{1-t}}\right),\end{aligned}$$

where ϕ is the standard normal density function. □

Details on Example 5

We assume that $\sigma(t,\mu(t)) \neq 0$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely and that $\mu(t)$, $e^{\Gamma(t)}$, $e^{-\Gamma(t)} \in \mathbb{D}_{1,2}$ with $D_t\,(\mu(t)) = \sigma(t,\mu(t))$ for all $t \in [0,T]$.[7]

Clearly, since $-mP_0$ is deterministic, the integrands of its MRT decomposition are zero. Thus:

$$R = L - \mathbb{E}\,(L) \;=\; \int_0^T (m - N(t-))e^{\Gamma(t)}\frac{\mathbb{E}\left(D_t\left(e^{-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right)}{\sigma(t,\mu(t))}dM^W(t) - \int_0^T \mathbb{E}\left(e^{\Gamma(t)-\Gamma(T)}\middle|\,\mathcal{G}_t\right)dM^N(t).$$

Since $D_t\,(\mu(s)) = 0$ for all $t > s$, and thus $D_t\,(\Gamma(t)) = 0$, the chain rule from Malliavin calculus

[7]Note that for globally Lipschitz-continuous coefficients θ and σ with at most linear growth, diffusion processes are Malliavin differentiable (Nualart, 2006, Theorem 2.2.1). However, as the discussion on the Malliavin differentiability of square-root processes shows (Alòs and Ewald, 2008), the general Malliavin differentiability of diffusion processes – and even affine processes – is not guaranteed.

(Nualart, 2006, Proposition 1.2.3) implies:

$$D_t\left(e^{\Gamma(t)-\Gamma(T)}\right) = -e^{\Gamma(t)-\Gamma(T)} D_t\left(\Gamma(T)-\Gamma(t)\right) = -e^{\Gamma(t)-\Gamma(T)} D_t\left(\Gamma(T)\right) = e^{\Gamma(t)} D_t\left(e^{-\Gamma(T)}\right),$$

i.e. $e^{\Gamma(t)}\mathbb{E}\left(D_t\left(e^{-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right) = \mathbb{E}\left(D_t\left(e^{\Gamma(t)-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right)$. Furthermore, exchanging conditional expectation and Malliavin derivative operator (Di Nunno et al., 2009, Proposition 3.12) together with (8) we have:

$$\mathbb{E}\left(D_t\left(e^{\Gamma(t)-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right) = D_t\left(\mathbb{E}\left(e^{\Gamma(t)-\Gamma(T)}\middle|\,\mathcal{G}_t\right)\right) = D_t\left(e^{\alpha(t)+\beta(t)\mu(t)}\right).$$

The chain rule finally yields:

$$D_t\left(e^{\alpha(t)+\beta(t)\mu(t)}\right) = e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)D_t\left(\mu(t)\right) = e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)\sigma(t,\mu(t)).$$

Hence, all-in-all, we obtain:

$$R = L - \mathbb{E}\left(L\right) = \int_0^T \left(m - N(t-)\right) e^{\alpha(t)+\beta(t)\mu(t)}\beta(t) dM^W(t) - \int_0^T e^{\alpha(t)+\beta(t)\mu(t)} dM^N(t).$$

For the second part, it is evident that the function f^A satisfies the smoothness condition in this case. In more general situations, one can for instance rely on the (sufficient) conditions in Heath and Schweizer (2000). Of course, in case an analytic expression cannot be determined, the respective function f can be computed numerically. □

Details on the Example from Section 4

Since r is an affine process, it follows that:

$$\mathbb{E}\left(e^{-\int_t^T r(s)ds}\middle|\,\mathcal{G}_t\right) = e^{\alpha_r(t,T)-\beta_r(t,T)r(t)},\ \ T\in[t,T^*],$$

where (Brigo and Mercurio, 2007, p. 66):

$$\alpha_r(t,T) = \tfrac{2\kappa\theta}{\sigma_r^2}\log\left(\tfrac{2he^{(\kappa+h)\frac{T-t}{2}}}{2h+(\kappa+h)(e^{h(T-t)}-1)}\right),\ \beta_r(t,T) = \tfrac{2(e^{h(T-t)}-1)}{2h+(\kappa+h)(e^{h(T-t)}-1)},\ h=\sqrt{\kappa^2+2\sigma_r^2}.$$

Similarly, the mortality intensity process is affine so that: $\mathbb{E}\left(e^{-\int_t^T \mu(s,x)ds}\middle|\,\mathcal{G}_t\right) = e^{\alpha_\mu(t,T,x)-\beta_\mu(t,T,x)\mu(t,x)}$, $T\in[t,T^*]$, where α_μ and β_μ satisfy the ordinary differential equations (ODEs) specified in Proposition 3.1 of Dahl and Møller (2006, p. 197). For deriving the MRT decomposition of $R = L - \mathbb{E}(L)$ with L defined in (9), first note that L can be rewritten as:

$$\begin{aligned} L = &\sum_{k=1}^{T} (m - N(t_{k-1}))\, e^{-\int_0^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\} \\ &- \sum_{k=1}^{T} (m - N(t_k))\, e^{-\int_0^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\}, \end{aligned} \tag{35}$$

i.e. it is a sum of survival benefits. We thus define the functions:

$$\begin{aligned} f_k^{A1}(t,x) &= \mathbb{E}\left(e^{-\int_t^{t_{k-1}} \mu(s)ds} e^{-\int_t^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\} \middle| X(t) = x \right), \;\; 0 \le t \le t_{k-1}, \\ f_k^{B1}(t,x) &= \mathbb{E}\left(e^{-\int_t^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\} \middle| X(t) = x \right), \;\; 0 \le t \le t_k, \\ f_k^{A2}(t,x) &= \mathbb{E}\left(e^{-\int_t^{t_k} [\mu(s)+r(s)]ds} \max\{P_0 - A(t_k), 0\} \middle| X(t) = x \right), \;\; 0 \le t \le t_k, \end{aligned}$$

which can be simplified by using the independence of S, r, and μ, as well as exploiting the log-normal distribution of S and the affine property of r and μ. This immediately shows that all three functions are sufficiently smooth, so that we can apply Proposition 4, part 2. We obtain the MRT decomposition:

$$R = L - \mathbb{E}(L) = R_{\text{fund}} + R_{\text{int}} + R_{\text{sys\_m}} + R_{\text{unsys\_m}}.$$

where the systematic risk component for fund (fund, $i = 1$), interest (int, $i = 2$), systematic mortality risk (sys_m, $i = 3$), and unsystematic mortality risk (unsys_m, $i = 4$) are given by:

$$\begin{aligned} R_i = &\sum_{k=1}^{T} \Bigg(\int_0^{t_{k-1}} (m - N(t-)) e^{-\int_0^t r(s)ds} \frac{\partial f_k^{A1}}{\partial x_i}(t, X(t)) dM_i^W(t) \\ &\quad + \int_{t_{k-1}}^{t_k} (m - N(t_{k-1})) e^{-\int_0^t r(s)ds} \frac{\partial f_k^{B1}}{\partial x_i}(t, X(t)) dM_i^W(t) \Bigg) \\ &- \sum_{k=1}^{T} \int_0^{t_k} (m - N(t-)) e^{-\int_0^t r(s)ds} \frac{\partial f_k^{A2}}{\partial x_i}(t, X(t)) dM_i^W(t), \end{aligned}$$

respectively, and the unsystematic mortality risk component is given by:

$$R_4 = -\sum_{k=1}^{T} \int_0^{t_{k-1}} e^{-\int_0^t r(s)ds} f_k^{A1}(t, X(t)) dM^N(t) + \sum_{k=1}^{T} \int_0^{t_k} e^{-\int_0^t r(s)ds} f_k^{A2}(t, X(t)) dM^N(t).$$

In our numerical calculations, we perform $N = 100,000$ simulations for determining the distributions of R, R_1, R_2, R_3, and R_4. For projecting the risk drivers r and μ as well as for approximating the stochastic integrals, we use an Euler scheme with $n = 100$ time steps per year. The number of survivors in the portfolio is projected by means of the binomial distribution conditioned on the mortality intensities. We solve the ODEs associated with the mortality model numerically using the Runge-Kutta method.

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