The Marginal Cost of Risk and Capital Allocation in a Property and Casualty Insurance Company

Qiheng Guo†  Daniel Bauer  George H. Zanjani

September 2021

Abstract

We develop a multi-period profit maximization model for a property and casualty (P&C) insurance company and use it for determining the marginal cost of risk for different lines of business and resulting economic capital allocations. In contrast to previous literature, our model features a loss structure that matches the characteristics of a P&C company, comprising so-called “short-tailed” and “long-tailed” business lines with different expected settlement terms. As one key contribution, our theoretical and numerical results show that lines with different terms are assessed differently, depending on the company’s financial situation.

JEL Classification: C61, G22, G32.
Keywords: loss triangle; chain-ladder; profit maximization; marginal cost of risk; capital allocation; NAIC data.

*We gratefully acknowledge funding from the Casualty Actuarial Society (CAS) under a Committee on Theory of Risk (COTOR) research project. The authors are grateful for helpful comments from Tim Boonen, Alietia Caughron, Richard Derrig, Cameron Ellis, Edward Frees, Michael Hoy, Ty Leverty, Dongchen Li, Junhao Liu, Michael Ludkovski, Lawrence Marcus, Lawrence McTaggart, Stephen Mildenhall, Ajay Subramanian, Ruilin Tian, Mary Weiss, Kenny Wunder, Huan Zhang, and seminar participants at the 2016 Insurance: Mathematics and Economics Congress (IME 2016), the 51st Actuarial Research Conference (ARC 2016), the American Risk and Insurance Association (ARIA) 2016, 2019 Annual Meeting, the UGA 2017 Ph.D. Research Symposium, Illinois State University, the University of California Santa Barbara, the University of St. Thomas, and the University of Wisconsin–Madison.
†Corresponding author. Phone: +1-(765)-285-8645. E-mail address: qguo@bsu.edu.
1 Introduction

Allocating capital to business lines is a key quantitative task in an insurance enterprise. Since the risk associated with different business lines has to be supported by capital and since capital is costly, capital allocation is essential for pricing and performance measurement in multi-line settings. Thus, it is no surprise to find a large number of papers on the subject in the actuarial and broader insurance literatures, although the focus is primarily on technical aspects of how to divide risk capital given by an arbitrary risk measure in a static environment (see e.g. Albrecht, 2004; Bauer and Zanjani, 2013; McNeil et al., 2015, Section 8.5 and references therein).

A distinguishing financial characteristic of P&C insurers is that business lines vary in the length of time it takes for claims to be reported and to settle.\footnote{Indeed, this is one of the primary aspects addressed in the vast actuarial literature on loss reserving in non-life insurance. We refer to the textbooks by Wüthrich and Merz (2008), Taylor (2012), and Radtke et al. (2016) for details.} The final resolution of the liabilities associated with a given policy can take years or even decades within so-called long-tailed lines such as general liability or workers’ compensation insurance. Moreover, even as time passes, significant uncertainties remain about the total payments that will ultimately be associated with any set of contracts. The situation leads to particular concerns for P&C insurers: In addition to the “shock” to liabilities from exposure to risk in the current accident year, companies are also exposed to “aftershocks” from exposure to risks taken in previous years. This impacts the dependence structure of the loss payments through time as well as the relevant information to be taken into account for assessing the financial situation of an insurer. As a consequence, loss development potentially affects optimal capitalization decisions and insurance portfolios.

Previous contributions on capital allocation in P&C resolve the mismatch between static capital allocation approaches based on risk measures and this dynamic nature of loss development in P&C insurance by focussing on the aggregate ultimate loss resulting from all lines of business (Brehm, 2002; Côté et al., 2016; Meyers, 2019; Rehman, 2016, among others). Hence, while details of P&C insurance are considered in modeling the ultimate losses, the approach relies a single loss variable for each line that is aggregated across time. In particular, the approach to capital allocation is not fundamentally different than for single-period exposures. This perspective is limiting in that it omits the possibility of adjusting future capitalization and underwriting decisions based on partial resolution of uncertainty. For instance, a high first year loss realization in a short-tailed line may be different than a high first year loss realization in a long-tailed line, as the latter may allows for adapting the company’s decisions before the realization of the ultimate loss.\footnote{Bauer et al. (2019) consider the uncertainty in the timing of different underwriting opportunities in constructing an insurance optimal portfolio subject to a risk measure. However, they do not consider the influence of the resolution of loss uncertainty in a dynamic context, which is the focus of this paper.}

To explore these aspects, we integrate a general P&C loss structure given via so-called loss triangles within a multi-period profit maximization model for an insurer that economizes on different capitalization options in an economic environment with financing frictions as in Bauer and Zanjani (2016, 2021). In this setting, economic capital allocations can be derived from the marginal cost of risk appearing in the company’s optimality conditions. However, the framework presented in Bauer and Zanjani (2016, 2021) is confined to single-period exposures and, therefore, is not suited for the specific circumstances of P&C insurer—where capital allocation is arguably most relevant. Providing and exploring such a framework are the key contributions of this paper.

In line with the results for single-period exposures, we find that while the marginal cost takes
the conventional form of the value of future liabilities plus allocated capital costs, the company evaluates uncertain liabilities under adjusted probabilities that reflect company effective risk aversion (Froot and Stein, 1998). However, we demonstrate that the adjustment differs for payments due in the current accident year versus payments in future development years, since the associated weights depend on the company’s (expected) financial situation. In particular, we document a difference between the firm’s aggregated marginal cost for new business and actually incurred (capital) costs in the current underwriting period, with the latter being affected by legacy exposures. This difference in treatment for liabilities with different durations implies differences in the assessment of shorter versus longer tailed business lines—and, thus, affects the company’s optimal liability structure. Both differences in the loss distribution as well as expected settlement times between business lines will interact with the financial situation of the company to determine their valuations.

To illustrate the trade-offs, we solve our model numerically in a setting with two business lines and two development periods for the long-tailed line (2L2DY) calibrated to US industry data using workers’ compensation and commercial automobile insurance exposures. More precisely, for the workers’ compensation line, we consider two development periods (long-tailed), where we assume that the losses develop according to a Chain-Ladder model with jointly normal innovations (Mack, 1993). For the commercial auto line we use a single development period (short-tailed). We implement the firm’s profit maximization problem by dynamic programming on a discretized state space. In line with Bauer and Zanjani (2021), we find that the value of the P&C insurer is concave with an optimal point that results from balancing profit expectations and capital costs. However, in our setting, we also find that the loss history has a considerable effect on the value function. The reason is transparent in this simple setting: Ideally, the company would optimize over period exposures to the new exposures (long-tailed and short-tailed) and the legacy exposure to the long-tailed line; however, the latter was fixed in the past, so the decision will be contingent on this past choice, and will also affect the company’s situation in the future. Our setting trades off these various consideration in a dynamic context.

Unsurprisingly, a company with a larger legacy exposure has a greater optimal capitalization level. Furthermore, the optimal exposure to the short-tailed line increases in company capital up to a saturation point, which is in line with earlier results for single-period exposures. In contrast and as a novel insight from this model with loss history, the optimal exposure to the long-tailed line is decreasing in capital and a company with a high legacy exposure will optimally underwrite more aggressively in the long-tailed line. The intuition is that the long-tailed line provides access to short-term financing that is valuable when the company is in bad financial shape.

These effects are reflected in the marginal cost of risk and resulting capital allocations—unlike in approaches presented in the actuarial literature or those relied upon in actuarial practice. In particular, the marginal cost features a term that evaluates the impact of expanding the long-tailed line on future firm value—whereas other cost components such as actuarial value or capital costs that also appear in other settings relate to risky indemnity payments in the current period. Hence, this term can be viewed as a cost of the financing opportunities the long-tailed line provides. Generally, the consideration of loss history is important in our calculations, and assessing performance via a risk-adjusted return ratio (RAROC) that does not take it into account appropriately will lead to inefficient outcomes. And while changes in the model setting may affect the form of the adjustment, the reasons leading to its genesis are fundamentally tied to the dynamic nature of the problem rather than our modeling choices.
The remainder of the paper is organized as follows. Section 2 presents a general model of multi-period profit maximization with a general loss structure in a P&C company. Section 3 derives the marginal cost of risk in this model and economic capital allocations. Section 4 presents an implementation of the 2L2DY model and a solution to the profit maximization problem. Section 5 presents marginal cost of risk and RAROC calculations. Finally, Section 5 concludes. All proofs and technical material, such as on numerical details and convergence of the employed algorithm, are relegated to an Online Appendix.

2 Multi-Period Profit Maximization with Loss History

2.1 Loss Structure for a P&C Company

Setting up a profit maximization framework for a P&C company requires modeling the asset and the liability sides. For simplicity, we assume the company’s assets bear no risk and that all the uncertainty originates from the liability side, modeled via claim payment amounts.

A P&C company writes new insurance contracts in each of its business lines at the beginning of every year (accident year), during which accidents occur and losses are reported. However, some of the losses are not reported until the next year or even years after the origination of the contract. Furthermore, only a portion of the payments is settled in the accident year, whereas the remainder of the (unrealized) payments will take several years to settle. The lags in reporting and paying losses are accounted for by considering so-called loss development years. Such a loss structure is typically represented via so-called loss triangles, with one triangle recording incurred (reported) losses, and another triangle recording paid losses. To illustrate, in Figure 1 we consider a P&C insurance company with \( N \) business lines with corresponding (paid) loss random variables \( L_{n,i}^{(1)} \) to \( L_{n,d_n}^{(d)} \), with line identifier \( n = 1, 2, \ldots, N \), accident year (AY) \( i = 1, 2, \ldots, t - d_n, \ldots, t, \ldots \), development year (DY) \( j = 1, 2, \ldots, d_n \), and \( i + j - 1 \) being the calendar year (period). For each variable, we only need to identify the development and calendar year and thus drop the accident year subscripts for simplicity.

In the paid loss triangle, for example, \( L_{1,1}^{(1)} \) to \( L_{d_n}^{(d_n)} \) denote amounts paid (if positive, or amount received if negative) for insurance sold at the beginning of year 1 in line \( n \). Thus, every year, there are payments for losses incurred in the current year, as well as for losses developed from previous years. Specifically, payments in the same calendar year consist of the diagonal entries in the paid loss triangle. For example, payments in calendar year \( t \) correspond to \( (L_{1,t}^{(1)}, L_{2,t}^{(1)}, \ldots, L_{d_n,t}^{(1)}) \), which are double-boxed inside Figure 1. \( L_{1,t}^{(1)} \) represents losses from the contract sold in period \( t \). Other losses \( (L_{2,t}^{(1)}, \ldots, L_{d_n,t}^{(1)}) \) are developed from previous years’ losses, which are in oval boxes and themselves make up a triangle in Figure 1. We denote this “historical” loss triangle at time \( t - 1 \) as \( \Delta^{(n,t-1)} = \{L_{j,t}, t - d_n + 1 \leq i \leq t - 1\} \), which contains (partial) loss information from \( t - d_n + 1 \) to \( t - 1 \). Denote \( L_{t}^{(n)} = (L_{2,t}^{(n)}, \ldots, L_{d_n,t}^{(n)}) \) as losses paid in year \( t \) that developed from \( \Delta^{(n,t-1)} \). To account for the loss development in each accident year, it is common to assume that the paid losses triangles have a Markov structure:

\[
\mathbb{P}(\Delta^{(n,t)} | \Delta^{(n,t-1)}, \Delta^{(n,t-2)}, \ldots, \Delta^{(n,1)}) = \mathbb{P}(\Delta^{(n,t)} | \Delta^{(n,t-1)}),
\]

Also, as is common, we assume independence across accident years. A Markov structure and the
indifference assumption together fit most of the loss reserving methods in the P&C industry. It is possible to relax the independence assumption and allow cross-sectional correlations between accident years, at the cost of more complex derivations.

Under independence and Markov assumptions, loss random variables in each accident year are related as follows:

$$P\left( L_{n,t}^{(j)} | h(L_{n,t-1}^{(n,t-j+1:t-1)}), \ldots, h(L_{n,t-j+1}^{(n,t-1)}) \right) = P\left( L_{n,t}^{(j)} | h(L_{n,t-j-1}^{(n,t-j+1:t-1)}) \right).$$

The losses in the $j^{th}$ development year only depend on the information of the same accident year and on a function $h$ of loss information on the previous development years. For example, in the most popular stochastic loss reserving method, the so-called Chain-Ladder approach (Mack, 1993), $h$ is the cumulative summation operation:

$$P\left( L_{n,t}^{(j)} | h(L_{1,j-1}^{(n,t-j+1:t-1)}), \ldots, h(L_{1,t-1}^{(n,t-1)}) \right) = P\left( L_{n,t}^{(j)} | \sum_{k=1}^{j-1} L_{k}^{(n,t-k)} \right).$$

### 2.2 A Multi-Line Multi-Period Profit Maximization Model

To fully describe the dynamic liabilities that the P&C company faces, we assume the following underwriting process: At the beginning of every period $t$, the insurer chooses to underwrite certain amounts in each line of business and charges premium $p_n^{(n,t)}$ in return. The underwriting decision corresponds to choosing an exposure parameter $q_n^{(n,t)}$. The losses will be realized over the development years, but the payments are always contingent on the exposure parameter and paid loss random variables. Also note that in each period, the total indemnity payment includes losses incurred and paid in the current year, as well as losses developed from the past years and to be paid in the current calendar year. Thus, for business line $n$ in period $t$, the indemnity payment can be
presented via the following function $\mathcal{I}^{(n)}(\cdot)$:

$$I^{(n,t)} = \mathcal{I}^{(n,t)} \left( \{q^{(n,t)}, L_1^{(n,t)}\}; \{Q^{(n,t-1)}, L^{(n,t)}\} \right),$$

where we assume $\mathcal{I}^{(n,t)} (\{q^{(n,t)}, 0\}, \{Q^{(n,t-1)}, 0\}) = 0$. $Q^{(n,t-1)}$ is the vector of exposure parameters associated with triangle $\Delta^{(n,t-1)}$ and losses $L^{(n,t)}$. In what follows, we will assume that indemnity payments are proportional to the exposure parameters:

$$I^{(n,t)} = q^{(n,t)} \times L_1^{(n,t)} + Q^{(n,t-1)} \times L^{(n,t)},$$

but generalizations are possible at the expense of a more cumbersome analysis (Frees, 2017; Mildenhall, 2017). We denote the aggregate period indemnity across business lines by $I^{(n,t)} = \sum_{n=1}^{N} I^{(n,t)}$.

The company collects the full premium $p^{(n,t)}$ at the beginning of each period $t$ on each line. The aggregate period premium is $p^{(n,t)} = \sum_{n=1}^{N} p^{(n,t)}$. The company can raise capital $B^{(t)} \geq 0$ or shed capital $B^{(t)} < 0$. The latter corresponds to paying dividends to shareholders. Since the company operates in an economic environment with financing frictions (Brunnermeier et al., 2012; Duffie, 2010), there is a cost of raising capital $c(B^{(t)})$ if $B^{(t)} \geq 0$, similarly as in Froot and Stein (1998) and Zanjani (2002). There is no cost of shedding capital, i.e. $c(B^{(t)}) = 0$ if $B^{(t)} < 0$. The company carries over capital $a^{(t-1)}(1 - \tau)$ from the last period, with $\tau$ denoting the unit frictional cost of internal capital. Raising external capital is always marginally more expensive than keeping internal capital, so we always have $c'(\cdot) > \tau > 0$.

Thus, the company’s assets at the beginning of period $t$ are:

$$a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(n,t)}.$$ 

During period $t$, the assets are invested at a fixed annual interest rate $r$. At the end of period $t$, the company pays the aggregate indemnity $I^{(n,t)}$ from its insurance policies sold in the current period and previous periods. The surplus of assets over aggregate indemnity, denoted by $a^{(t)}$, can then be carried over to period $t + 1$. Thus, we have the following law of motion for the company’s capital:

$$a^{(t)} = \left( a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(n,t)} \right) (1 + r) - I^{(n,t)}, \quad (1)$$

assuming $a^{(t)} \geq 0$. If the company defaults, it pays out all remaining assets to policyholders. The company cannot shed more capital than it has available. Hence, for $a^{(t-1)} \geq 0$, we require that:

$$B^{(t)} \geq -a^{(t-1)}(1 - \tau). \quad (2)$$

The objective function for each period can be derived using the revenue (premium collected), minus the costs (indemnity, frictional costs on carrying capital, and financing costs). For each

---

3In the rest of the paper, we use $X^{(n,t)}$ as the sum across the lines $\sum_{n=1}^{N} X^{(n,t)}$. $X^{(n,t)}$ is used to represent the line-by-line collection (vector) $(X^{(n,t)}, \ldots, X^{(N,t)})$, and its subset $X^{(m:n,t)} = (X^{(m,t)}, \ldots, X^{(n,t)})$. $X^{(n,t,t+s)}$ represents a collection of random variables over discrete time $(X^{(n,t)}, X^{(n,t+1)}, \ldots, X^{(n,t+s)})$.
period, the expected aggregate indemnity takes the following form:
\[ e^{(t)} = \mathbb{E} \left[ I^{(t)} \mathbf{1}_{\{a^{(t)} \geq 0, \ldots, a^{(t)} \geq 0\}} + (a^{(t)} + I^{(t)}) \mathbf{1}_{\{a^{(t)} \geq 0, \ldots, a^{(t)} < 0\}} | \Delta^{(t-1)} \right]. \]

Note that here we write the remaining assets in case of default as \( a^{(t)} + I^{(t)} < I^{(t)} \).

Hence, the company’s period profit function \( f \) is:
\[
\begin{align*}
    f(s_t) &= \{a^{(t-1)}, Q^{(t-1)}, \Delta^{(t-1)}\}, c_t = \{q^{(t)}, p^{(t)}, B^{(t)}\} \\
    &= (1 + r)p^{(t)} - e^{(t)} - (1 + r)(\tau a^{(t-1)} + c(B^{(t)})) \\
    &= \mathbb{E} \left[ \mathbf{1}_{\{a^{(t)} \geq 0, \ldots, a^{(t)} \geq 0\}} \{ (1 + r)p^{(t)} - I^{(t)} - (1 + r)(\tau a^{(t-1)} + c(B^{(t)})) \} \\
    &\quad - \mathbf{1}_{\{a^{(t)} \geq 0, \ldots, a^{(t)} < 0\}} (1 + r)(a^{(t-1)} + B^{(t)}) | s_t \right].
\end{align*}
\]

The “state” \( s_t \) contains all the variables that determine the state of the company at the beginning of period \( t \). The “control” \( c_t \) contains all the variables that the company chooses in maximizing the objective function, also at the beginning of period \( t \). In particular, both \( s_t \) and \( c_t \) are predictable with the information from the loss triangle \( \Delta^{(t-1)} \). The insurance company’s ultimate objective is to maximize future expected discounted cash flows, which corresponds to the following infinite horizon optimization problem:
\[
\max_{c_t} \sum_{t=1}^{\infty} \mathbb{E}[\beta^t f(s_t, c_t)],
\]
where \( \beta = (1 + r)^{-1} \) is the discount factor. The objective function can be equivalently represented as present value of future dividends as follows:
\[
\max_{c_t} \mathbb{E} \left[ \sum_{t \leq t^*} -\beta^{t-1} B^{(t)} - a^{(0)} \right],
\]
where \( t^* \) is the time such that \( a^{(1)} \geq 0, a^{(2)} \geq 0, \ldots, a^{(t^*-1)} \geq 0, a^{(t^*)} < 0 \) (see Lemma A.1 Appendix A for the proof).

We solve the optimization with constraints (1), (2), a premium function for each line \( n \), and a regulatory constraint if needed. For the premium function, we follow Bauer and Zanjani (2021) and assume that the premium charged for one line is the expected present (actuarial) value of future losses multiplied by a markup function. The present value of future losses for each line at the end of period \( t \) can be represented as
\[
R^{(n,t)} = \sum_{j=1}^{d_n} \beta^{j-1} q^{(n,t)} L_j^{(n,t+j-1)}.
\]

The markup function is a (decreasing) function of company risk \( \phi \) and size \( \theta = \mathbb{E} [R^{(t)} | \Delta^{(t-1)}] \), defined as
\[
\pi^{(n)} = \pi^{(n)}(\phi, \theta),
\]
with the assumption on partial derivatives:
\[ \pi_1^{(n)} = \frac{\partial \pi^{(n)}(\phi, \theta)}{\partial \phi} < 0, \quad \text{and} \quad \pi_2^{(n)} = \frac{\partial \pi^{(n)}(\phi, \theta)}{\partial \theta} < 0, \]
so a company with greater risk and larger size charges a smaller markup over actuarial value.

\( \phi \) is a risk metric that measures the risk of a company given its total indemnities and assets. For measuring “risk,” we assume that the policyholders are concerned with the company’s period solvency, so that the risk depends on total indemnities paid \( I^{(t)} \) and total end-of-periods assets \( S^{(t)} \):
\[ \phi = \phi(I^{(t)}, S^{(t)}), \]
where \( S^{(t)} = a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(t)}(1 + r) \). Here, similarly to Bauer and Zanjani (2021), in addition to obvious monotonicity assumptions \( \phi(I, x) \leq \phi(I, y), x \geq y, \) and \( \phi(X, x) \leq \phi(Y, x), X \leq Y, \) we assume scale invariance of the risk metric, i.e. \( \phi(aI, ax) = \phi(I, x), a > 0. \)

The key example that we will rely on in our numerical applications is the conditional default probability:
\[ \phi(I^{(t)}, S^{(t)}) = \mathbb{P}(I^{(t)} > S^{(t)} | \Delta^{(t-1)}). \]
We note that this specification assumes consumers are myopic in that they are only concerned with the coming period—and not necessary the performance of their contract down the line. This may be justified with the assumption that consumers rely on company ratings that obviously do not depend on the term of the obligation. Also, using the risk metric will correspond to risk measures over single year exposures that are common in the industry (for instance, using the default probability will correspond to a one-year Value-at-Risk). However, we flag that this assumption is relevant for some of the results that follow, although of course it is possible to swap out the premium function for an alternative and develop alternative results accordingly by following our same procedure.

Altogether, we have the following premium function for line \( n \):
\[ p^{(n,t)} = \mathbb{E} \left[ \beta R^{(n,t)} | \Delta^{(t-1)} \right] \times \pi^{(n)}(\phi, \theta). \] (6)

According to Bertsekas (1995), the optimization problem (4) is an infinite-horizon discrete-time stochastic optimal control problem, resulting in the following Bellman equation:

**Proposition 2.1. (Bellman Equation).** The Bellman equation for problem (4) reads:
\[
\begin{align*}
V(a^{(t-1)}, Q^{(t-1)}, \Delta^{(t-1)}) &= \max_{q^{(t-1)}, p^{(t-1)}, B^{(t-1)}} \mathbb{E} \left[ I_{\{I^{(t-1)} \leq S^{(t-1)}\}} \left( p^{(t-1)} - \beta I^{(t-1)} - \tau a^{(t-1)} - c(B^{(t-1)}) + \beta V(a^{(t)}, Q^{(t)}, \Delta^{(t)}) \right) \\
&\quad - I_{\{I^{(t)} > S^{(t-1)}\}} \left( a^{(t-1)} + B^{(t-1)} \right) | \Delta^{(t-1)} \right],
\end{align*}
\]
subject to (1), (2), and (6)

Here the default threshold for the company is \( S^{(t)} \). Once the aggregate indemnity is greater than \( S^{(t)} \), the company defaults. We do not consider the option of raising emergency capital to save the company as in Bauer and Zanjani (2021), since the focus of this paper is on how loss history, i.e. past exposures \( Q^{(t-1)} \) and losses \( \Delta^{(t-1)} \), affect the optimal exposure, raising, and allocation decisions. However, incorporating emergency capital is theoretically straightforward.
In particular, when incorporating emergency raising capital, we note that the model in Bauer and Zanjani (2021) will be a special case of the general model here with one development year in all business lines, thus effectively reducing the value function to one dimension with \( a^{(t-1)} \). In our general setting, with the company having \( N \) lines and each line \( n \) having \( d_n \) development years, there are a total of \( 1 + \frac{1}{2} \sum_{n=1}^{N} (d_n^2 + d_n - 2) \) state variables.

### 3 Marginal Cost of Risks

Following Bauer and Zanjani (2021), we determine the income when marginally increasing exposure to line \( n \in \{1, \ldots, N\} \) at the portfolio optimum, which by the optimality condition will correspond to the marginal cost of risk. In particular, this cost features marginal capital costs for the exposure in the line, which can be used to derive economically motivated capital allocations for purposes of pricing and performance measurement.

To make headway, it is helpful to appreciate that “risk” in the current setting enters the company’s calculus through the risk functional \( \phi \): Policyholders worry about the risk in their exposure, and this is reflected in the premium markup \( \pi^{(n)} \) they are willing to pay. It has long been appreciated that this approach via counterparty risk aversion is one way of micro-founding capital allocation in the company’s optimization problem (Zanjani, 2002), with an alternate and largely equivalent approach imposing an exogenous risk measure constraint (McNeil et al., 2015, and references therein). In particular, the risk functional \( \phi \) yields an allocation of company capital \( S^{(t)} \):

**Lemma 3.1.** The company’s period capital \( S^{(t)} \) can be allocated to the company’s exposures according to gradients of the risk functional:

\[
S^{(t)} = \sum_{i=1}^{N} \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\partial}{\partial q^{(i,t-j)}} \phi(I^{(t)}, S^{(t)}) - \frac{\partial}{\partial S^{(t)}} \phi(I^{(t)}, S^{(t)}) \frac{\partial}{\partial q^{(i,t-j)}} .
\]  

(7)

Here different risk metrics yield allocations according to different homogeneous risk measures \( \rho \). For instance, as indicated above, choosing \( \phi \) as the probability of default \( \phi = \mathbb{P}(I^{(t)} > S^{(t)}) \) yields an allocation according to Value-at-Risk, choosing \( \phi = \mathbb{E}\left[(I^{(t)}/S^{(t)} - 1) \mathbf{1}_{\{I^{(t)}>S^{(t)}\}}\right] \) yields Expected Shortfall as the allocation risk measure, etc. (we refer to Section 2.1 in Bauer and Zanjani (2021) for more details). It is worth emphasizing that in line with previous literature, capital is a shared resource that is allocated to exposures corresponding to different lines of business \( (1 \leq i \leq N) \) incurred at different points in time \( (t - d_n < j \leq t) \).

Armed with this insight, we rely on the first-order conditions of the Bellman equation in Proposition 2.1 to derive the marginal cost of risk:
Proposition 3.1. (Marginal Cost of Risks Equation). We have for the marginal cost of risk:

\[
MR_n = \mathbb{E} \left[ \sum_{j=0}^{d_n-1} \beta^j L_{j+1}^{(n,t+j)} | \Delta^{(:,t-1)} \right] \times \pi^{(n)} \left( 1 + \sum_{i=1}^{N} \frac{\pi_2^{(i)}}{\pi^{(n)}} \mathbb{E} \left[ R^{(i,t)} | \Delta^{(:,t-1)} \right] \right)
\]

\[
= \sum_{j=0}^{d_n-1} \mathbb{E} \left[ I_{\{I^{(t)} \leq S^{(t)}, \ldots, I^{*(t+s)} \leq S^{*(t+s)}\}} w_{t+j} (1 - c'(B^{(t)})) \beta^j L_{j+1}^{(n,t+j)} | \Delta^{(:,t-1)} \right]
\]

\[
+ \frac{\partial \rho(I^{(t)})}{\partial q^{(n,t)}} \mathbb{E} \left[ I_{\{I^{(t)} > S^{(t)}\}} w_t | \Delta^{(:,t-1)} \right],
\]

where:

\[
w_t = \begin{cases} 1 + V_1(a^{(t)}, Q^{(:,t)}, \Delta^{(:,t)}) & I^{(t)} \leq S^{(t)} \\ \sum_{i=1}^{N} \mathbb{E} \left[ R^{(i,t)} | \Delta^{(:,t-1)} \right] \times \frac{\pi_1^{(i)}}{\pi^{(n)}} \frac{\partial \phi}{\partial \Pi(I^{(t)}>S^{(t)})} & I^{(t)} > S^{(t)}, \end{cases}
\]

with \( \mathbb{E} \left[ (1 - c'(B^{(t)}))w_t | \Delta^{(:,t-1)} \right] = 1 \), and

\[
w_{t+s} = 1 + V_1(a^{*(t+s)}, Q^{*(t+s)}, \Delta^{*(t+s)})
\]

on \( I^{(t)} \leq S^{(t)}, \ldots, I^{*(t+s)} \leq S^{*(t+s)} \) from time \( t \) to \( t + d_n - 1 \), where \( a^{*(t+s)}, Q^{*(t+s)} \) etc. are the (stochastic) future state variables under the optimal policy.

The first line of Equation (8) represents the marginal premium income when marginally expanding business line \( n \), which is adjusted down due to the effect of company size on market prices (the last term reflecting scale costs). At the optimum, the marginal premium income will equal the marginal cost of risk.

A central result of this proposition, echoing the finding in Bauer and Zanjani (2021), is that the multi-period setting with loss history recovers the conventional form of the marginal cost of risk composed of (I) actuarial cost in solvent states plus (II) allocated capital costs. More precisely, the first expression on the right-hand side of the equation (I) consists of the valuation of future losses for accident year \( t \) in business line \( n \), namely losses \( (L_1^{(n,t)}, L_2^{(n,t+1)}, \ldots, L_{d_n}^{(n,t+d_n-1)}) \), which evolve according to a stochastic loss reserving model. However, and again in analogy to Bauer and Zanjani (2021), the valuation is modulated by functions \( w \) that put different weights on different loss states. The second term on the right-hand side of the equation represents the frictional (capital) costs for line \( n \), given by the capital allocation \( \frac{\partial \rho(I^{*(t)})}{\partial q^{(n,t)}} \) multiplied by associated capital costs, which take the form of a weighted default probability.

The state weights originate from company effective risk aversion. A marginal increase in risk \( (n) \) will produce changes in end-of-period outcomes, which will affect the value of the company—that is, there is a (random) cost associated with the continuation value of the company \( V_1 \). The factor \( (1 - c') \) reflects the fact that in the multi-period model, premiums act as a substitute for capital raised and thus save the company the marginal cost of raising capital. The weight in default states, similarly, follows from the sensitivity of the premium income to a change in the default threshold. Importantly, the weighting function integrates to one: The marginal value of raising one
extra dollar to the company is exactly one dollar, otherwise the company would raise more. This mechanism is familiar from asset pricing theory, where cash flows are weighted using marginal utilities of the representative consumer.

What is different in our setting relative to the single-period contracts in Bauer and Zanjani (2021) is that now the exposures extend beyond the next period, so that the valuation reflects payments in future periods \( t+1, t+2, \ldots, t+d_n - 1 \), until the losses are fully developed. By the same reasoning as before, since the company is effectively risk-averse, the valuation is not risk-neutral but reflects the company’s state valuation by using weights \( w_{t+s} = 1 + V_1(a^{(t+s)}, Q^{(t+s)}, \Delta^{(t+s)}) \).

However, and this is a key insight of our model, these state weights are different for different periods as they relate to the company valuation then—i.e., in period \( t+s, 1 \leq s < d_n \). In particular, depending on its situation, the company may attach different (state) valuations to liabilities in the near future and in the far future—and pinpointing those is relevant for choosing exposures in the current period.

To illustrate the latter point, we can reformulate Equation (8) by determining the Risk-Adjusted Return on Capital (RAROC):

\[
\text{RAROC}_n = \frac{\text{MR}_n - \sum_{j=0}^{d_n-1} \mathbb{E} \left[ \mathbf{1}_{\{I(t) \leq S(t) \} \times \Delta^{(t)} \leq \Delta^{(t+j)}} w_{t+j} \left( 1 - c'(B(t)) \right) \beta j I^{(n,t+j)} \mid \Delta^{(t-1)} \right]}{\sum_{j=0}^{d_n-1} \mathbb{E} \left[ \mathbf{1}_{\{I(t) > S(t) \}} w_t \mid \Delta^{(t-1)} \right]}.
\]

As is familiar, RAROC evaluates marginal income minus marginal liability values in solvent states, divided by allocated capital, and at the optimum all line-specific RAROCs align with marginal capital costs \( \mathbb{E} \left[ \mathbf{1}_{\{I(t) > S(t) \}} w_t \mid \Delta^{(t-1)} \right] \). The company can therefore compose its liability portfolio by gauging and comparing line RAROCs relative to a “hurdle rate.” However, to obtain accurate RAROCs, in the numerator the company needs to determine liability valuations reflecting adequate state weights, across different future periods.

A key difference to the case with single-period exposures is that total calendar-year costs and total marginal costs are no longer aligned. While for a scale-invariant risk metric per Lemma 3.1 total period capital \( S^{(t)} \) can be allocated to all exposures that affect solvency risk—with a cost charge amounting to the “hurdle rate”—these include past exposures with \( j > 0 \) in Equation (7). This is not surprising as these historical exposures do consume capital. In contrast, the marginal cost equations (8) and (9) only include charges for capital allocated to the “new” exposures \( \frac{\partial p(l^{(n,t)})}{\partial q(a,t)} \).

However, the marginal cost equations include the valuation of liabilities incurred in future calendar years—and those are exactly the terms with \( j > 0 \) in Equations (8) and (9). These are included with state weights \( w_{t+j} \) that account for their future consumption of capital. Hence, in contrast with the single-period exposures, even though we have a capital allocation that adds up in each period, the marginal costs do not add up to period costs but span period costs over several years.

To explore the interplay between these different components and particularly the state weights across periods, in the next section, we provide an implementation of our model in the context of a P&C insurer with two business lines and two development years.

\footnote{Return on Risk-Adjusted Capital (RORAC) and Risk-Adjusted Return on Risk-Adjusted Capital (RARORAC) are also used, with the differences not being consistently defined. We are agnostic with regards to the differences and utilize “RAROC” as a catch-all term for a return on capital measure that has been adjusted for risk.}
4  Application in P&C Insurance: A 2L2DY Setting

In this section, we implement our theory in the previous section in the context of a P&C insurer with two business lines and two development years on the long-tailed line (2L2DY). We then calibrate and solve for the model using numerical methods. Finally, we determine resulting capital allocations and line RAROCs, where we focus on exploring the impact of loss history on optimal portfolio decisions.

4.1 Implementation in a Chain-Ladder Model

In our 2L2DY setting, Line 1 is the long-tailed line that develops for two years, so that the paid loss triangle is a 2x1 triangle, as illustrated in Figure 2. Line 2 is assumed to be the short-tailed line with no development years beyond the accident year. The time period equals to AY + DY - 1. Therefore, at the end of current period $t$, the insurer faces losses $L_{1,1}^{(t)}$ and $L_{1,2}^{(t)}$ from its long-tailed line 1, and $L_{2,1}^{(2,t)}$ from its short-tailed line 2. The loss random variables above the solid lines in Figure 2 are realized before $t$. The grayed-out $L_{1,2}^{(1,t+1)}$ is not a part of the loss triangle and not realized until the end of the next period $t + 1$, but it is relevant to the premium written for the accident year $t$ and therefore related to the insurer’s problem.

We simplify the notation by denoting $L_{1}^{(n,t-2+j)}$ as $L_{1}^{(n)}j$ and $L_{1}^{(n,t-1+j)}$ as $L_{1}^{(n)}j$ with a prime “′” denoting state variables in the next period. See the illustration in Figure 3. We put three assumptions on our loss triangles: (i) The losses in Line 1 develop according to a Chain-Ladder model; (ii) we assume the losses are follow a (conditional) normal distribution; and (iii) the line-losses exhibit a constant correlation. These assumptions make the model tractable and allow us to efficiently calculate the moments of loss random variables in the Bellman equation. Details of three distributional assumptions are presented in Online Appendix B.

In the Bellman equation, now the past exposure state variables $Q(\cdot, t-1) = q(1)$ and past losses state variables $\Delta(\cdot, t-1) = L_{1}^{(1)}$. Hence, the optimization problem from Proposition 2.1 now takes the following form:

$$V(a, q^{(1)}, L_{1}^{(1)}) = \max_{q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)}, B} \beta \mathbb{E} \left[ 1_{\{I \leq S\}} (S - I) + 1_{\{I \leq S\}} V(a', q^{(1)}, L_{1}^{(1)}) \right] - a - B,$$
Marginal Cost of Risk and Capital Allocation in a P&C Insurance Company

<table>
<thead>
<tr>
<th>Line 1</th>
<th>Line 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>DY</td>
<td></td>
</tr>
<tr>
<td>AY</td>
<td>1</td>
</tr>
<tr>
<td>t-1</td>
<td>$L_1^{(1)}$</td>
</tr>
<tr>
<td>t</td>
<td>$L_1^{(1)}$</td>
</tr>
<tr>
<td></td>
<td>$L_1^{(2)}$</td>
</tr>
</tbody>
</table>

Figure 3: Losses relevant to the Bellman equation for a company under 2L2DY

where:

$$S = (a(1 - \tau) + B - c_1(B) + p^{(1)} + p^{(2)})(1 + r),$$

$$a' = S - I,$$

$$I = q^{(1)}L_1^{(1)} + q^{(2)}L_1^{(2)} + q^{(1)}L_2^{(1)}$$

For the premium functions, we assume the following specification:

$$p_n = \mathbb{E}\left[\beta R^{(n)}\right] \times \exp \left\{\alpha_n - \delta_n \mathbb{P}(I > S) - \gamma_n \mathbb{E}[R]\right\}, n = 1, 2,$$

where $R^{(1)} = q^{(1)}L_1^{(1)} + \beta q^{(1)}L_2^{(1)}$, $R^{(2)} = q^{(2)}L_1^{(2)}$, and $R = R^{(1)} + R^{(2)}$ representing the aggregate risk in the premium $p_1 + p_2$. Note that the aggregate risk $R$ does not equal to the aggregate indemnity $I$, because of the long-tailed line. $I$ reflects losses to be paid out in the current time period, while $R$ entails risks exposed in one accident year across two periods. $\mathbb{E}[R]$ reflects the aggregate scale of the insurance business. As indicated before, the motivation for this specification is that policyholders assess company quality via ratings that reflect the default probability ($\ldots - \delta_n \mathbb{P}(I > S)\ldots$), and increasing the scale of insurance business decreases profit margins ($\ldots - \gamma_n \mathbb{E}[R]\ldots$).

In the numerical implementation of 2L2DY model, we need to choose the premium parameters, loss triangle parameters, and company level parameters. For fixing premium parameters, we run the regression:

$$\log p_{it} = \alpha + \alpha_t + \delta d_{it} + \gamma E_{it} + \epsilon_{it},$$

where $p_{it}$ is the ratio of net premium earned to the sum of loss paid, net loss adjustment and underwriting expenses; $d_{it}$ denotes the default probability; $E_{it}$ is the expected loss, a measure of the size of the company, calculated using the net premium earned multiplied by the average loss ratio; and $\epsilon_{nit}$ is an error term.\(^5\) We collect the net premium collected, loss paid, loss adjustment costs, and underwriting costs from 2004 to 2013 NAIC statutory data for select lines.\(^6\) To obtain the default probability, we use the one-year default probability in Exhibit 2 under Best’s Impairment Rate and Rating Transition Study - 1977-2014 with proper interpolation. We round the regression

\(^5\)In this paper, we choose the same premium parameters for both lines. Although a generalization to two distinct sets of premium parameters is possible, it complicates the model solution and may yield results that are difficult to interpret. We leave corresponding extensions for future research.

\(^6\)The National Association of Insurance Commissioners (NAIC) is the U.S. standard-setting and regulatory support organization created and governed by the chief insurance regulators from the 50 states, the District of Columbia and five U.S. territories.
results for simplicity, but the magnitudes are representative.

For fixing the numbers in our loss reserving model, we use the paid loss triangles of workers’ compensation and commercial automobile from a representative P&C insurance company, obtained from NAIC Schedule P data from 2013. To suit the 2L2DY model, we only use paid losses of the accident year and first development year for the long-tailed line. In the short-tailed line, we use the one-year treasury constant maturity rate from the FRED Economic Data to discount the first development year loss to the accident year. Then we estimate the chain-ladder and normal parameters. Again, we round the results to have a set of simple parameters.

For the capital cost parameters, we set $\tau = 0.03$, $c_1^{(1)} = 0.075$, $c_1^{(2)} = 1.0 \times 10^{-10}$, and the risk-free interest rate is $r = 0.03$, as in the “base case” scenario in Bauer and Zanjani (2021). The complete set of parameters is listed in Table 1. A solution to the 2L2DY model and corresponding numerical techniques are detailed in Online Appendix C.

### 4.2 Numerical Results

The value function and optimal policies are functions of three state variables: current assets $a$, past exposure to the long-tailed line $q^{(1)}$, and historic loss realization in the long-tailed line $L_1^{(1)}$ (see Equation (10)), so it is impossible to illustrate them in a single graph. Therefore, we illustrate our results in two separate figures with two extreme levels of the previous shock $L_1^{(1)}$ (large and small). More precisely, Figure 4 uses a small previous shock of two standard deviations below the mean at $L_1^{(1)} = 5.0E7$, whereas Figure 5 assumes a large previous shock of two standard deviations above the mean at $L_1^{(1)} = 1.5E8$. Since all panels figures are qualitatively similar but with distinct levels, the two levels provide a good overview of the solution. Each figure includes results on the value
function $V$ (first line), the optimal capitalization decision $B$ (second line), the optimal exposure to the long-tailed line this period $q^{(1)}$ (third line), and the optimal exposure to the short-tailed line this period $q^{(2)}$ (fourth line). For each quantity, we present a three-dimensional illustration via a heat map (first column), a two-dimensional illustration as a function of assets $a$ for different choices of past exposures $q^{(1)}$ (second column), and a two-dimensional illustration as a function of $q^{(1)}$ for different choices of $a$ (third column).

The value function, optimal raising of external capital, and exposure to the short-tailed line all match the corresponding characteristics in Bauer and Zanjani (2021). More precisely, the value function is concave with an optimal capitalization level that economizes on costly external financing, internal capital costs, and an optimal company size (see panels (b)). The firm raises capital if it is underfunded or sheds capital (pays dividends) if it is overfunded, but remains inactive for capitalization levels around the optimal point (see the flat part in between the negatively sloped segments in panels (e)). The optimal exposure to the short-tailed line is increasing in the capital level, up to a saturation point where costs associated with scale do not warrant further expansion (panels (k)).

However, the solution here additionally provides insights on how previous exposure in the long-tailed line affect the value function and optimal policies. The value function decreases in past exposure, as is obvious from panels (c) of the figures. Furthermore, by comparing the figures for different past exposures in panels (b), and by comparing the two panels between the figures, we find that the optimal capitalization point increases in both the previous exposure $q^{(1)}$ and the loss realization $L_1^{(1)}$. This is not surprising since capital needs increase in both.

The optimal long-tailed line exposure is depicted in Figures 4(g)-(i) and 5(g)-(i). Interestingly, the optimal exposure is increasing with previous exposure—which may be counter-intuitive at first sight. Also, in most situations as capital increases, an insurer sells more insurance in the short-tailed line and raises less external capital, but sells less in the long-tailed line. The reason is that the insurer will be paying the full indemnity of the loss incurred in the short-tailed line, but only a fraction of the loss incurred in the long-tailed line, while it earns full premium on both lines. As a result, the long-tailed line offers an alternate source of short-term financing for a firm in need of funds. This effect is more pronounced for a high previous exposure, since more short-term financing is needed.

An exception is the low exposure ($q^{(1)} = 0.1$) in the small previous shock scenario, where exposure to the long-tailed line increases in capital just as for the short-tailed line. Hence, the long-tailed line can take two functions, depending on the company’s capitalization situation: either it can be a exposure or it can take the role of a funding source. For higher capital levels a saturation point with an optimal mix between the short- and long-tailed exposures is reached.

5 Marginal Cost of Risk and RAROC in the 2L2DY Setting

In order to appraise the relevance of multi-period exposures on optimal decisions and capital allocations in our model, we present the marginal cost of risk and capital allocation for our 2L2DY setting.
Figure 4: Illustration of the optimal solution for a small previous shock of $L_1^{(1)} = 5.0E7$. First line: Value function $V$. Second line: Optimal external capital raising $B$. Third line: Optimal exposure to long-tailed line $q_1^{(1)}$. Fourth line: Optimal exposure to short-tailed line $q_2^{(2)}$. All as functions of total assets $a$ and past exposure to long-tailed line $q_1^{(1)}$. 
Figure 5: Illustration of the optimal solution for a large previous shock of $L_1^{(1)} = 1.5E8$. First line: Value function $V$. Second line: Optimal external capital raising $B$. Third line: Optimal exposure to long-tailed line $q^{(1)}$. Fourth line: Optimal exposure to short-tailed line $q^{(2)}$. All as functions of total assets $a$ and past exposure to long-tailed line $q^{(1)}$. 
5.1 Marginal Cost Equation

With the assumptions and notation simplifications described in the previous section, the marginal cost of risk equation (8) takes the following form:

**Proposition 5.1.** *(Marginal Cost of Risk for 2L2DY)* We have:

\[
MR_n = \left( \mathbb{E}[R_q^{(n)}] \times \exp \left( \alpha - \delta \mathbb{P}(I > S) - \gamma \mathbb{E}[R] \right) \right) \times (1 - \gamma \mathbb{E}[R])
\]

\[
= \frac{\mathbb{E}\left[ (1 - c_1(B)) R_q^{(n)} I_{1 \leq S} \right]}{(i)} + \mathbb{E}\left[ I_q^{(n)} (1 - c_1(B)) V_1 I_{1 \leq S} \right] \quad (ii)
\]

\[
+ \frac{\partial V}{\partial q} \mathbb{P}(I > S) + \tau^* + \frac{\partial V}{\partial q} \left[ (1 - c_1(B)) \left( -V_2 - \beta L_2^{(n)} \right) I_{1 \leq S & n=1} \right] \quad (iv)
\]

where \( I_q^{(n)} = \partial I^{(i)}/\partial q^{(n)}, \quad R_q^{(n)} = \partial R^{(n)}/\partial q^{(n)}, \quad V_1 \) is the derivative of \( V \) in its first dimension (capital), and \( V_2 \) is the derivative of \( V \) in its second dimension (past exposure). Therefore, \( V_2 \) is non-zero in the long-tailed Line 1 only, and \( \partial V/\partial q^{(i)} = -I_q^{(1)} V_1 + V_2 \). The Value-at-Risk is evaluated at \( \psi = \mathbb{P}(I \leq S) \), and \( \tau^* = \frac{\partial c_1(B)}{1 - c_1(B)} - \mathbb{E}[V_1 I_{1 \leq S}] \) is the cost of capital net benefits of capital in solvent states.

In line with the general case in equation (8), the left-hand side of the equation (11) is the marginal premium income, which at the optimum marginal premium income equals the marginal cost of risk. The right-hand side decomposes the marginal cost of risk in line \( n \) into four components. The first three are: (i) Expected actuarial costs in solvent states. (ii) Consequences of expansion on the continuation value of the company. And (iii) capital costs associated with the exposures. In contrast to equation (8), here we split up the valuation of the period liabilities into parts (i) and (ii) by expanding the period-weight \( w_t \), which also gives us the explicit form of the hurdle rate in (iii). These components also arise in a setting with single-period exposures, and Bauer and Zanjani (2021) document that they can be connected to the company’s *effective risk* influencing how the company values uncertain cash flows (Froot and Stein, 1998).

The final component (iv) is a novel feature and is associated with the company’s liability structure. It arises in equation (8) from the weight associated with the future exposure, \( w_{t+1} \). More precisely, in the long-tailed line, the underwriting decision today will impact not only losses incurred in the current period, but will also lead to “legacy” liabilities in next year’s optimization problem. As is evident from panels (c) in Figures 4 and 5, \( V_2 \) generally is negative and concave for high past exposure. Therefore, this “aftershock” component (iv) which takes the marginal cost of legacy liabilities in future value in excess of their actuarial cost \( \mathbb{E}[\beta L_2^{(1)}] \) generally is positive, i.e. it is a “true” cost. However, of course this cost is balanced by the additional premium income the company receives now, namely \( \mathbb{E}[\beta L_2^{(1)}] \times \exp(\alpha - \delta \mathbb{P}(I > S) - \gamma \mathbb{E}[R]). \) In other words, this component can be interpreted as a financing cost for premium capital that is not directly linked to servicing current-year indemnities, to which the other components (i)-(iii) correspond.

While—or because—the marginal cost equation therefore includes component (iv) that is not directly linked to costs incurred this year, it also does not reflect the total (capital) cost incurred this
year. Indeed, while an aggregation of marginal costs from equation (11) reflects capital allocations for current exposures $q^{(1)}$ and $q^{(2)}$ when allocating:

$$ S = \text{VaR}_\psi(I) = \text{VaR}_\psi(q^{(1)}L_1^{(1)} + q^{(2)}L_1^{(2)} + q^{(1)}L_2^{(1)}) $$  \hspace{1cm} (12)

there is also an allocation to the legacy exposure $q^{(1)}$ (see also Lemma 7). This legacy exposure was fixed in the past when contemplating an optimal portfolio, i.e., it was reflected in a previous marginal cost equation—based on a different information set. As a consequence, legacy exposure affects capital allocations and optimal portfolios. We will investigate these interrelations on the context of the calibrated version of our model in what follows.

### 5.2 Numerical Results

We present three sets of analyses that establish the relevance of legacy exposures to the company’s liability costs. First, we decompose the total marginal cost of risk, $q^{(n)} \times MR_n$, for both lines into the four components (i)-(iv), in order to gauge the relevance of the “aftershock” component (iv). We also determine resulting RAROCs as in equation (9) with considering different components on the cost side. In particular, we analyze whether more conventional RAROC calculations that omit the novel components (ii) and (iv) can present close approximations for guiding pricing and performance measurement.

Finally, we decompose the current year’s total cost into (i) the expected period costs consisting of the expected actuarial cost of indemnity payments in solvent states, (ii) the expected impact on the on firm value, and (iii) capital costs for the current-year exposures of the short- and long-tailed lines plus (iv)' the current-year exposure originating from legacy liabilities in the short-tailed line. In particular, this second exercise will include a complete allocation of period capital costs as in equation (12).

Again, we consider different scenarios for capital levels $a$—zero, medium ($a = 200,000,000$), and high ($a = 1,000,000,000$); for past exposures—low ($q^{(1)} = 0.1$) and high ($q^{(1)} = 0.7$); and legacy loss realization—small ($L_1^{(1)} = 50,000,000$) and large ($L_1^{(1)} = 150,000,000$) as in Section 4.2. Table 2 through 5 present the results.

### Decomposition of the Marginal Cost of Risk

Table 2 through 4 show the marginal cost decompositions for different capital levels—Table 2 for low, Table 3 for medium, and Table 4 for high capital level. For each table, we show results for low and high past exposures in the top and bottom parts, for a small (left) and large (right) loss realization. Since for the low value $q^{(1)}$ the past loss realization are less relevant, it is not surprising that the numbers for the left- and right-hand sides of the top part of each table are similar. Several of the results are in line with the setting with single-period exposures from Bauer and Zanjani (2021)—particularly for the short-tailed line. For lower capital levels, capital costs (iii) present a greater portion of the total costs. The financial impact of expanding business on firm value is relatively small and even negative for large capital levels due to costly internal capital. Indeed, it only exceeds a percentage point for zero capital ($a = 0$) for the short-tailed line.
Marginal Cost of Risk and Capital Allocation in a P&C Insurance Company

\[ \alpha = 0 \]

<table>
<thead>
<tr>
<th>Low ( q^{(1)} )</th>
<th>Small ( L_i^{(1)} )</th>
<th>Large ( L_i^{(1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>Line 2</td>
<td>Line 1</td>
</tr>
<tr>
<td>(i) Actuarial Cost</td>
<td>197,081,652</td>
<td>99,884,284</td>
</tr>
<tr>
<td></td>
<td>85.81%</td>
<td>82.2%</td>
</tr>
<tr>
<td>(ii) Continuation Value</td>
<td>1,861,561</td>
<td>1,774,203</td>
</tr>
<tr>
<td></td>
<td>0.81%</td>
<td>1.46%</td>
</tr>
<tr>
<td>(iii) Capital Cost</td>
<td>22,333,044</td>
<td>19,860,529</td>
</tr>
<tr>
<td></td>
<td>9.72%</td>
<td>16.34%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>221,276,257</td>
<td>121,519,016</td>
</tr>
<tr>
<td></td>
<td>96.34%</td>
<td>100%</td>
</tr>
<tr>
<td>(iv) Aftershock</td>
<td>8,398,544</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3.66%</td>
<td>0%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>229,674,801</td>
<td>121,519,016</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High ( q^{(1)} )</th>
<th>Small ( L_i^{(1)} )</th>
<th>Large ( L_i^{(1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>Line 2</td>
<td>Line 1</td>
</tr>
<tr>
<td>(i) Actuarial Cost</td>
<td>196,532,417</td>
<td>99,874,671</td>
</tr>
<tr>
<td></td>
<td>86.18%</td>
<td>84.48%</td>
</tr>
<tr>
<td>(ii) Continuation Value</td>
<td>1,849,343</td>
<td>1,715,109</td>
</tr>
<tr>
<td></td>
<td>0.81%</td>
<td>1.45%</td>
</tr>
<tr>
<td>(iii) Capital Cost</td>
<td>18,433,698</td>
<td>16,627,990</td>
</tr>
<tr>
<td></td>
<td>8.08%</td>
<td>14.07%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>216,815,458</td>
<td>118,217,770</td>
</tr>
<tr>
<td></td>
<td>95.07%</td>
<td>100%</td>
</tr>
<tr>
<td>(iv) Aftershock</td>
<td>11,240,845</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>4.93%</td>
<td>0%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>228,056,304</td>
<td>118,217,770</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 2: Marginal costs of risk under low capital level

In contrast, the novel component \((iv)\) that does not arise for single-period exposures makes up a non-trivial portion of the total costs besides the actuarial costs. It is positive and amounts to around four percent of total marginal cost in the long-tailed line for the low capital scenario, and to about two percent for the medium- and highly capitalized scenario.

The relevance of this component is also reflected in risk-adjusted return calculations. From Equation (11), the risk-adjusted return on capital (RAROC) for line \(i\) can be written as the ratio of marginal profit and economic capital allocated:

\[
RAROC_i = \frac{MR_i - [(i) + (ii) + (iv)]}{\partial VaR_\psi(I) / \partial q^{(n)}} = (\mathbb{P}(I > S) + \tau^*)
\]

We note that conventional RAROC calculations do not account for components \((ii)\) and \((iv)\), so we analyze how their omission changes resulting risk-adjusted returns.

Table 5 displays several RAROC calculations in different situations (capital levels, past exposures) and accounting for different cost components. More precisely, the “Correct Allocation” includes all cost components; “w/o Aftershock” omits component \((iv)\); “Act. Only” omits compo-
### Table 3: Marginal costs of risk under medium capital level

<table>
<thead>
<tr>
<th>$a = 200,000,000$</th>
<th>Small $L_i^{(1)}$</th>
<th>Large $L_i^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low $q^{(1)}$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(i) Actuarial Cost</td>
<td>196,739,583</td>
<td>99,554,150</td>
</tr>
<tr>
<td></td>
<td>94.96%</td>
<td>93.1%</td>
</tr>
<tr>
<td>(ii) Continuation Value</td>
<td>455,648</td>
<td>402,181</td>
</tr>
<tr>
<td></td>
<td>0.22%</td>
<td>0.38%</td>
</tr>
<tr>
<td>(iii) Capital Cost</td>
<td>6,795,081</td>
<td>6,976,613</td>
</tr>
<tr>
<td></td>
<td>3.28%</td>
<td>6.52%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>203,990,313</td>
<td>106,932,944</td>
</tr>
<tr>
<td></td>
<td>98.46%</td>
<td>100%</td>
</tr>
<tr>
<td>(iv) Aftershock</td>
<td>3,189,949</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1.54%</td>
<td>0%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>207,180,262</td>
<td>106,932,944</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a = 200,000,000$</th>
<th>Small $L_i^{(1)}$</th>
<th>Large $L_i^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High $q^{(1)}$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(i) Actuarial Cost</td>
<td>196,903,165</td>
<td>99,618,580</td>
</tr>
<tr>
<td></td>
<td>95.83%</td>
<td>95.63%</td>
</tr>
<tr>
<td>(ii) Continuation Value</td>
<td>355,426</td>
<td>305,374</td>
</tr>
<tr>
<td></td>
<td>0.17%</td>
<td>0.29%</td>
</tr>
<tr>
<td>(iii) Capital Cost</td>
<td>4,304,787</td>
<td>4,247,092</td>
</tr>
<tr>
<td></td>
<td>2.1%</td>
<td>4.08%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>201,563,379</td>
<td>104,171,046</td>
</tr>
<tr>
<td></td>
<td>98.1%</td>
<td>100%</td>
</tr>
<tr>
<td>(iv) Aftershock</td>
<td>3,898,117</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1.9%</td>
<td>0%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>205,461,496</td>
<td>104,171,046</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>
Table 4: Marginal costs of risk under high capital level

<table>
<thead>
<tr>
<th></th>
<th>Small $L_i^{(1)}$</th>
<th></th>
<th>Large $L_i^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Line 1</td>
<td>Line 2</td>
<td>Line 1</td>
</tr>
<tr>
<td></td>
<td>94.97%</td>
<td>93.11%</td>
<td>94.89%</td>
</tr>
<tr>
<td>Continuation</td>
<td>469,223</td>
<td>417,292</td>
<td>437,794</td>
</tr>
<tr>
<td></td>
<td>0.23%</td>
<td>0.39%</td>
<td>0.21%</td>
</tr>
<tr>
<td>Capital Cost</td>
<td>6,794,185</td>
<td>6,975,693</td>
<td>7,075,804</td>
</tr>
<tr>
<td></td>
<td>3.28%</td>
<td>6.5%</td>
<td>3.41%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>204,183,426</td>
<td>107,378,003</td>
<td>204,229,092</td>
</tr>
<tr>
<td></td>
<td>98.48%</td>
<td>100%</td>
<td>98.52%</td>
</tr>
<tr>
<td>Aftershock</td>
<td>3,159,923</td>
<td>0</td>
<td>3,070,993</td>
</tr>
<tr>
<td></td>
<td>1.52%</td>
<td>0%</td>
<td>1.48%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>207,343,348</td>
<td>107,378,003</td>
<td>207,300,086</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Small $L_i^{(1)}$</th>
<th></th>
<th>Large $L_i^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Line 1</td>
<td>Line 2</td>
<td>Line 1</td>
</tr>
<tr>
<td>Actuarial Cost</td>
<td>196,364,743</td>
<td>99,820,663</td>
<td>196,844,317</td>
</tr>
<tr>
<td></td>
<td>95.78%</td>
<td>95.64%</td>
<td>97.33%</td>
</tr>
<tr>
<td>Continuation</td>
<td>367,704</td>
<td>305,664</td>
<td>-1,008,858</td>
</tr>
<tr>
<td></td>
<td>0.18%</td>
<td>0.29%</td>
<td>-0.5%</td>
</tr>
<tr>
<td>Capital Cost</td>
<td>4,304,503</td>
<td>4,246,812</td>
<td>3,393,140</td>
</tr>
<tr>
<td></td>
<td>2.1%</td>
<td>4.07%</td>
<td>1.68%</td>
</tr>
<tr>
<td>(i)-(iii) Total</td>
<td>201,036,950</td>
<td>104,373,139</td>
<td>199,228,599</td>
</tr>
<tr>
<td></td>
<td>98.05%</td>
<td>100%</td>
<td>98.51%</td>
</tr>
<tr>
<td>Aftershock</td>
<td>3,989,736</td>
<td>0</td>
<td>3,005,450</td>
</tr>
<tr>
<td></td>
<td>1.95%</td>
<td>0%</td>
<td>1.49%</td>
</tr>
<tr>
<td>(i)-(iv) Total</td>
<td>205,026,687</td>
<td>104,373,139</td>
<td>202,234,050</td>
</tr>
<tr>
<td></td>
<td>100.00%</td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>
### Table 5: RAROC calculations

<table>
<thead>
<tr>
<th></th>
<th>( a = 0 )</th>
<th>( a = 200,000,000 )</th>
<th>( a = 1,000,000,000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Small ( L_1^{(1)} )</strong></td>
<td>( a = 0 )</td>
<td>( a = 200,000,000 )</td>
<td>( a = 1,000,000,000 )</td>
</tr>
<tr>
<td>Correct Allocation</td>
<td>10.49% 10.49% 12.02% 12.02%</td>
<td>14.43% 10.49% 19.35% 12.02%</td>
<td>15.31% 11.43% 20.55% 13.26%</td>
</tr>
<tr>
<td>w/o Aftershock</td>
<td>14.43% 10.49% 19.35% 12.02%</td>
<td>15.31% 11.43% 20.55% 13.26%</td>
<td>16.03% 10.59% 21.28% 14.11%</td>
</tr>
<tr>
<td>Act. Only</td>
<td>15.31% 11.43% 20.55% 13.26%</td>
<td>16.03% 10.59% 21.28% 14.11%</td>
<td>17.39% 12.18% 21.67% 14.41%</td>
</tr>
<tr>
<td><strong>Large ( L_1^{(1)} )</strong></td>
<td>( a = 0 )</td>
<td>( a = 200,000,000 )</td>
<td>( a = 1,000,000,000 )</td>
</tr>
<tr>
<td>Correct Allocation</td>
<td>10.59% 10.59% 14.11% 14.11%</td>
<td>16.03% 10.59% 21.28% 14.11%</td>
<td>17.39% 12.18% 21.67% 14.41%</td>
</tr>
<tr>
<td>w/o Aftershock</td>
<td>16.03% 10.59% 21.28% 14.11%</td>
<td>17.39% 12.18% 21.67% 14.41%</td>
<td>18.25% 13.01% 24.59% 15.34%</td>
</tr>
<tr>
<td>Act. Only</td>
<td>17.39% 12.18% 21.67% 14.41%</td>
<td>18.25% 13.01% 24.59% 15.34%</td>
<td>19.22% 13.46% 25.63% 15.88%</td>
</tr>
</tbody>
</table>
components \((ii)\) and \((iv)\).

By design, the “correct allocation” equalizes RAROC for both lines under the optimized portfolio. In line with the results from Bauer and Zanjani (2021), ignoring cost components relating to firm value as for the “Act.” results distorts the balance and usually inflates the absolute value of the RAROC—since cost components are ignored. However, the impact of ignoring the component \((ii)\) is relatively minor, since it amounts to a relatively small fraction of total marginal costs. In contrast, since the “Aftershock” cost component \((iv)\) now accounts for a non-trivial share of the total cost, it is not surprising that ignoring it will lead to distortions with RAROC ratios. In particular, it makes the line look overly profitable, because we effectively do not account for relevant capital costs for the exposure in future years.

**Decomposition of the Period Risk Costs**

To contrast the previous decomposition of firm marginal costs for new business, Tables 6 through 8 presents an allocation of total costs incurred in the current calendar year. Here we include cost components originating from past exposures, including a total allocation of capital (costs) as shown in equation (12).

No surprisingly, the legacy costs make up for larger portions for high past exposures \(q^{(1)}\) and a large realization of \(L^{(1)}_1\). A key point here is that optimal exposures to current-year risks both in the long and short-tailed line will respond to the legacy exposure. For instance, the components associated with the short-tailed line 2 are small when capital is lower and legacy exposures is high, simply because these are exactly the situations where the long-tailed line is more attractive due to its function as a source of short-term financing. In contrast, for a well-capitalized firm with low legacy exposure, the firm is in a good position to take on risk exposure this period. Hence, the short-tailed line is attractive and makes up for the largest costs component, although the company also enters into line 1 for risk diversification.

### 6 Conclusion

We develop a model for capital allocation for a P&C insurance company operating in a dynamic setting. The general model is very flexible and can be applied to insurance companies that have both short-tailed and long-tailed business lines. The implementation of the model with two lines and two development years shows previous loss exposure and loss realization significantly affect the company’s optimal policies. We find that long-tailed lines are employed as short-term sources of financing, an insight that considerably changes the characteristics and optimal policies relative to short-tailed lines.

Various extensions are possible. First, for tractability, we adopt a chain-ladder method with normal distributions. In the actuarial literature, there are more advanced models for estimating and forecasting loss triangles that may be considered. Second, we begin with the assumption that all the assets are invested at a fixed interest rate. Adding securities markets to the general model as well as other financing options or reinsurance would further bridge the gap between model and reality.
### Marginal Cost of Risk and Capital Allocation in a P&C Insurance Company

#### Table 6: Total Calendar Year Cost Allocation – Low Capitalization ($a = 0$)

<table>
<thead>
<tr>
<th>a = 0 Low $q^{(1)}$ and Small $L^{(1)}$</th>
<th>Line 1 Current</th>
<th>Line 2 Current</th>
<th>Line 1 Legacy</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>27,866,711</td>
<td>12,460,713</td>
<td>4,944,384</td>
<td>45,271,809</td>
</tr>
<tr>
<td>$E [I_{I_i \leq S}]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>518,921</td>
<td>221,334</td>
<td>85,455</td>
<td>825,710</td>
</tr>
<tr>
<td>$E [IV_1 I_{I_i \leq S}]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>6,225,466</td>
<td>2,477,631</td>
<td>1,211,165</td>
<td>9,914,261</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>34,611,098</td>
<td>15,159,678</td>
<td>6,241,004</td>
<td>56,011,780</td>
</tr>
</tbody>
</table>

| Low $q^{(1)}$ and Large $L^{(1)}$      |                |                |               |       |
| Actuarial Value of Solvent Payments (i) | 35,522,995     | 4,363,412      | 14,978,464    | 54,864,871 |
| $E [I_{I_i \leq S}]$                   |                |                |               |       |
| $\Delta$ Company Valuation of Solvent Payment (ii) | 945,021        | 110,830        | 372,343       | 1,428,194 |
| $E [IV_1 I_{I_i \leq S}]$             |                |                |               |       |
| Capital Cost (iii)                     | 7,336,982      | 738,809        | 3,190,369     | 11,266,160 |
| $S (P(I > S) + \tau^*)$               |                |                |               |       |
| Agg. Marginal Cost (i)-(iii)           | 43,804,998     | 5,213,051      | 18,541,176    | 67,559,225 |

| High $q^{(1)}$ and Small $L^{(1)}$     |                |                |               |       |
| Actuarial Value of Solvent Payments (i) | 46,180,925     | 10,463,609     | 34,872,624    | 91,517,158 |
| $E [I_{I_i \leq S}]$                   | 50.46%         | 11.43%         | 38.11%        | 100%  |
| $\Delta$ Company Valuation of Solvent Payment (ii) | 856,488        | 179,687        | 761,140       | 1,797,316 |
| $E [IV_1 I_{I_i \leq S}]$             | 47.65%         | 10%            | 42.35%        | 100%  |
| Capital Cost (iii)                     | 8,537,215      | 1,742,071      | 14,509,478    | 24,788,764 |
| $S (P(I > S) + \tau^*)$               | 34.44%         | 7.03%          | 58.53%        | 100%  |
| Agg. Marginal Cost (i)-(iii)           | 55,754,628     | 12,385,367     | 50,143,242    | 118,103,238 |

| High $q^{(1)}$ and Large $L^{(1)}$     |                |                |               |       |
| Actuarial Value of Solvent Payments (i) | 49,457,717     | 99,879         | 104,770,285   | 154,327,880 |
| $E [I_{I_i \leq S}]$                   | 32.05%         | 0.06%          | 67.89%        | 100%  |
| $\Delta$ Company Valuation of Solvent Payment (ii) | 261,084        | 381            | 954,724       | 1,216,189 |
| $E [IV_1 I_{I_i \leq S}]$             | 21.47%         | 0.03%          | 78.5%         | 100%  |
| Capital Cost (iii)                     | 9,558,899      | 17,637         | 36,697,534    | 46,274,071 |
| $S (P(I > S) + \tau^*)$               | 20.66%         | 0.04%          | 79.3%         | 100%  |
| Agg. Marginal Cost (i)-(iii)           | 59,277,700     | 117,897        | 142,422,543   | 201,818,140 |

| $\Delta$ Company Valuation of Solvent Payment (ii) | 518,921        | 221,334        | 85,455        | 825,710 |
| $E [IV_1 I_{I_i \leq S}]$             | 62.85%         | 26.81%         | 10.35%        | 100%  |
| Capital Cost (iii)                     | 6,225,466      | 2,477,631      | 1,211,165     | 9,914,261 |
| $S (P(I > S) + \tau^*)$               | 62.79%         | 24.99%         | 12.22%        | 100%  |
| Agg. Marginal Cost (i)-(iii)           | 34,611,098     | 15,159,678     | 6,241,004     | 56,011,780 |

### Notes:
- $a = 0$ represents the low capitalization scenario.
- $q^{(1)}$ and $L^{(1)}$ denote the risk and capital allocation levels, respectively.
- The table provides a detailed breakdown of actuarial values, company valuations, capital costs, and aggregated marginal costs for different risk scenarios and capitalization levels.
<table>
<thead>
<tr>
<th>$a = 200,000,000$</th>
<th>Line 1 Current</th>
<th>Line 2 Current</th>
<th>Line 1 Legacy</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low $q^{(1)}$ and Small $L_1^{(1)}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>32,954,856</td>
<td>37,672,648</td>
<td>4,994,706</td>
<td>75,622,211</td>
</tr>
<tr>
<td>$E[I_{I \leq S}]$</td>
<td>43.58%</td>
<td>49.82%</td>
<td>6.6%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>150,374</td>
<td>152,191</td>
<td>15,238</td>
<td>317,803</td>
</tr>
<tr>
<td>$E[IV_{I \leq S}]$</td>
<td>47.32%</td>
<td>47.89%</td>
<td>4.79%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>2,242,531</td>
<td>2,640,045</td>
<td>295,027</td>
<td>5,177,603</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>43.31%</td>
<td>50.99%</td>
<td>5.7%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>35,347,761</td>
<td>40,464,884</td>
<td>5,304,971</td>
<td>81,117,617</td>
</tr>
<tr>
<td><strong>Low $q^{(1)}$ and Large $L_1^{(1)}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>32,884,661</td>
<td>27,952,694</td>
<td>14,965,227</td>
<td>75,802,582</td>
</tr>
<tr>
<td>$E[I_{I \leq S}]$</td>
<td>43.38%</td>
<td>36.88%</td>
<td>19.74%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>145,974</td>
<td>106,477</td>
<td>47,832</td>
<td>300,283</td>
</tr>
<tr>
<td>$E[IV_{I \leq S}]$</td>
<td>48.61%</td>
<td>35.46%</td>
<td>15.93%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>2,326,994</td>
<td>1,923,036</td>
<td>969,608</td>
<td>5,219,638</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>44.58%</td>
<td>36.84%</td>
<td>18.58%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>35,357,629</td>
<td>29,982,207</td>
<td>15,982,667</td>
<td>81,322,503</td>
</tr>
<tr>
<td><strong>High $q^{(1)}$ and Small $L_1^{(1)}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>34,712,796</td>
<td>30,095,478</td>
<td>34,778,795</td>
<td>99,587,069</td>
</tr>
<tr>
<td>$E[I_{I \leq S}]$</td>
<td>34.86%</td>
<td>30.22%</td>
<td>34.92%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>123,471</td>
<td>92,256</td>
<td>249,751</td>
<td>465,477</td>
</tr>
<tr>
<td>$E[IV_{I \leq S}]$</td>
<td>26.53%</td>
<td>19.82%</td>
<td>53.65%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>1,495,436</td>
<td>1,283,077</td>
<td>3,559,430</td>
<td>6,337,943</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>23.59%</td>
<td>20.24%</td>
<td>56.16%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>36,331,703</td>
<td>31,470,811</td>
<td>38,587,976</td>
<td>106,390,489</td>
</tr>
<tr>
<td><strong>High $q^{(1)}$ and Large $L_1^{(1)}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>45,853,964</td>
<td>3,303,816</td>
<td>104,461,555</td>
<td>153,619,335</td>
</tr>
<tr>
<td>$E[I_{I \leq S}]$</td>
<td>29.85%</td>
<td>2.15%</td>
<td>68%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>-37,251</td>
<td>-6,753</td>
<td>304,300</td>
<td>260,296</td>
</tr>
<tr>
<td>$E[IV_{I \leq S}]$</td>
<td>-14.31%</td>
<td>-2.59%</td>
<td>116.91%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>1,582,202</td>
<td>105,108</td>
<td>6,776,112</td>
<td>8,463,423</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>18.69%</td>
<td>1.24%</td>
<td>80.06%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>47,398,915</td>
<td>3,402,171</td>
<td>111,541,967</td>
<td>162,343,054</td>
</tr>
<tr>
<td>Table 7: Total Calendar Year Cost Allocation – Medium Capitalization ($a = 200,000,000$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low $q^{(1)}$ and Small $L_1^{(1)}$</td>
<td>Line 1 Current</td>
<td>Line 2 Current</td>
<td>Line 1 Legacy</td>
<td>Total</td>
</tr>
<tr>
<td>------------------------------</td>
<td>--------------</td>
<td>--------------</td>
<td>--------------</td>
<td>-------</td>
</tr>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>32,972,141</td>
<td>37,835,694</td>
<td>4,984,856</td>
<td>75,792,690</td>
</tr>
<tr>
<td>$E [I I_{I \leq S}]$</td>
<td>43.5%</td>
<td>49.92%</td>
<td>6.58%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>154,854</td>
<td>157,909</td>
<td>16,327</td>
<td>329,090</td>
</tr>
<tr>
<td>$E [IV_1 I_{I \leq S}]$</td>
<td>47.06%</td>
<td>47.98%</td>
<td>4.96%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>2,242,235</td>
<td>2,639,697</td>
<td>294,988</td>
<td>5,176,920</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>43.31%</td>
<td>50.99%</td>
<td>5.7%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>35,369,230</td>
<td>40,633,300</td>
<td>5,296,171</td>
<td>81,298,700</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Low $q^{(1)}$ and Large $L_1^{(1)}$</th>
<th>Line 1 Current</th>
<th>Line 2 Current</th>
<th>Line 1 Legacy</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>32,835,558</td>
<td>27,978,495</td>
<td>14,961,805</td>
<td>75,775,858</td>
</tr>
<tr>
<td>$E [I I_{I \leq S}]$</td>
<td>43.33%</td>
<td>36.92%</td>
<td>19.74%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>143,985</td>
<td>106,194</td>
<td>45,045</td>
<td>295,224</td>
</tr>
<tr>
<td>$E [IV_1 I_{I \leq S}]$</td>
<td>48.77%</td>
<td>35.97%</td>
<td>15.26%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>2,327,151</td>
<td>1,923,198</td>
<td>969,677</td>
<td>5,220,026</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>48.77%</td>
<td>35.97%</td>
<td>15.26%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>35,306,694</td>
<td>30,007,887</td>
<td>15,976,527</td>
<td>81,291,108</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High $q^{(1)}$ and Small $L_1^{(1)}$</th>
<th>Line 1 Current</th>
<th>Line 2 Current</th>
<th>Line 1 Legacy</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>34,627,353</td>
<td>30,156,529</td>
<td>34,894,857</td>
<td>99,678,739</td>
</tr>
<tr>
<td>$E [I I_{I \leq S}]$</td>
<td>34.74%</td>
<td>30.25%</td>
<td>35.01%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>127,737</td>
<td>92,343</td>
<td>254,481</td>
<td>474,561</td>
</tr>
<tr>
<td>$E [IV_1 I_{I \leq S}]$</td>
<td>26.92%</td>
<td>19.46%</td>
<td>53.62%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>1,495,337</td>
<td>1,282,992</td>
<td>3,559,195</td>
<td>6,337,524</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>23.59%</td>
<td>18.58%</td>
<td>56.16%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>36,250,427</td>
<td>31,531,864</td>
<td>38,708,533</td>
<td>106,490,824</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High $q^{(1)}$ and Large $L_1^{(1)}$</th>
<th>Line 1 Current</th>
<th>Line 2 Current</th>
<th>Line 1 Legacy</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Value of Solvent Payments (i)</td>
<td>36,950,986</td>
<td>21,268,632</td>
<td>104,063,786</td>
<td>162,283,404</td>
</tr>
<tr>
<td>$E [I I_{I \leq S}]$</td>
<td>22.77%</td>
<td>13.11%</td>
<td>64.12%</td>
<td>100%</td>
</tr>
<tr>
<td>$\Delta$ Company Valuation of Solvent Payment (ii)</td>
<td>-372,986</td>
<td>-226,862</td>
<td>-676,907</td>
<td>-1,276,755</td>
</tr>
<tr>
<td>$E [IV_1 I_{I \leq S}]$</td>
<td>29.21%</td>
<td>17.77%</td>
<td>53.02%</td>
<td>100%</td>
</tr>
<tr>
<td>Capital Cost (iii)</td>
<td>1,254,482</td>
<td>700,789</td>
<td>6,807,891</td>
<td>8,763,162</td>
</tr>
<tr>
<td>$S (P(I &gt; S) + \tau^*)$</td>
<td>14.32%</td>
<td>8%</td>
<td>77.69%</td>
<td>100%</td>
</tr>
<tr>
<td>Agg. Marginal Cost (i)-(iii)</td>
<td>37,832,482</td>
<td>21,742,559</td>
<td>110,194,770</td>
<td>169,769,811</td>
</tr>
</tbody>
</table>

Table 8: Total Calendar Year Cost Allocation – High Capitalization ($a = 1,000,000$)
References


Appendix

A Proofs

Lemma A.1. The optimization problem (4) can be equivalently represented as a maximization of the present value of future dividends:

$$\max_{c_t} \mathbb{E} \left[ \sum_{t \leq t^*} -\beta^{t-1} B(t) - a(0) \right],$$

where $t^*$ is the time such that $a^{(1)}(t^*) \geq 0, a^{(2)}(t^*) \geq 0, \ldots, a^{(t^*-1)}(t^*) \geq 0, a^{(t^*)}(t^*) < 0$.

Proof. The capital motion equation (1) can be rewritten into:

$$\beta^t a(t) - \beta^{t-1} a(t-1) - \beta^{t-1} B(t) = \beta^t \left[ (1 + r)p(t) - I(t) - \beta^t(1 + r)\tau a^{(t-1)} + c(B(t)) \right].$$

We can rewrite the objective function in (4) as the following:

$$\sum_{t=1}^{\infty} \mathbb{E} \left[ 1\{a^{(1)}(t) \geq 0, \ldots, a^{(t)}(t) \geq 0\} \beta^t \left\{ (1 + r)p(t) - I(t) - \beta^t(1 + r)\tau a^{(t-1)} + c(B(t)) \right\} - 1\{a^{(1)}(t) \geq 0, \ldots, a^{(t)}(t) < 0\} \beta^t(1 + r)\tau a^{(t-1)} + c(B(t)) \right],$$

Proof of Proposition 2.1: Since our objective function in (4) is bounded from above, following Bertsekas (1995), the infinite horizon optimization problem (4) subject to (1) is exactly resulting in the Bellman equation (2.1).

Proof of Lemma 3.1. The scale invariance property of $\phi$ yields (Bauer and Zanjani, 2021):

$$0 = \frac{\partial}{\partial w} \phi(wT(t), wS(t)) = S(t) \frac{\partial}{\partial S(t)} \phi(wT(t), wS(t)) + I(t) \frac{\partial}{\partial I(t)} \phi(wT(t), wS(t))$$

$$\Rightarrow S(t) = \sum_{i=1}^{N} \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\partial}{\partial q^{(i,t-j)}} \phi(T(i,t), S(t)) - \frac{\partial}{\partial S(t)} \phi(T(i,t), S(t)).$$
Proof of Proposition 3.1. The Bellman equation reads:

\[
V(a^{(t-1)}, Q^{(t-1)}, \Delta^{(t-1)}) = \max_{q^{(t)}, p^{(t)}, B^{(t)}} \mathbb{E}
\left[
1_{\{f^{(t)} \leq S^{(t)}\}} \left(p^{(t)} - \beta f^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, Q^{(t)}, \Delta^{(t)})\right) - 1_{\{f^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) | \Delta^{(t-1)} \right],
\]

where,

\[
I^{(t)} = \sum_{n=1}^{N} (q^{(n,t)} L_{1}^{(n,t)} + Q^{(n,t-1)} L^{(n,t)}),
\]

\[
S^{(t)} = (a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(t)}) (1 + r),
\]

\[
R^{(n,t)} = \sum_{j=1}^{d_{n}} \beta^{j-1} q^{(n,t)} R_{j}^{(n,t+j-1)},
\]

subject to:

\[
a^{(t)} = S^{(t)} - I^{(t)},
p^{(n,t)} = \mathbb{E} \left[ \beta R^{(n,t)} | \Delta^{(t-1)} \right] \pi^{(n)}(\phi, \theta), \quad n = 1, 2, \ldots, N.
\]

The premium function is the product of conditional expected present value of \(R^{(n,t)}\), or future losses of the accident year \(t\), and a markup function \(\pi^{(n)}\). The markup function consists of two arguments: a risk metric \(\phi\) and company size. We assume the risk metric is scale invariant, or \(\phi(w f^{(t)}, w S^{(t)}) = \phi(f^{(t)}, S^{(t)})\), \(w > 0\). Denote \(\frac{\partial \pi^{(n)}(x,y)}{\partial x} = \pi^{(1)}_{1}\) and \(\frac{\partial \pi^{(n)}(x,y)}{\partial y} = \pi^{(2)}_{n}\).

In addition, we denote the gradients of the value function:

\[
V_{1}(a, Q^{(t)}, \Delta^{(t)}) = \lim_{\delta \to 0} \frac{V(a + \delta, Q^{(t)}, \Delta^{(t)}) - V(a, Q^{(t)}, \Delta^{(t)})}{\delta}.
\]

For \(s = 1, 2, \ldots, d_{n} - 1\):

\[
V_{2,s}^{(n)}(a, Q^{(t)}, \Delta^{(t)}) = \lim_{\delta \to 0} \frac{V_{1}(a, Q_{1:n-1}^{(t)}, \delta Q_{n}^{(t)} + \delta, Q_{n+1:N}^{(t)}, \Delta^{(t)}) - V_{1}(a, Q_{1:n-1}^{(t)}, Q_{n}^{(t)}, Q_{n+1:N}^{(t)}, \Delta^{(t)})}{\delta}.
\]

The Lagrangian writes:

\[
\mathcal{L}^{(t)} = \mathbb{E}
\left[
1_{\{f^{(t)} \leq S^{(t)}\}} \left(p^{(t)} - \beta f^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, Q^{(t)}, \Delta^{(t)})\right) - 1_{\{f^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) | \Delta^{(t-1)} \right] - \sum_{i=1}^{N} \lambda^{(i,t)} \left(p^{(i,t)} - \mathbb{E} \left[ \beta R^{(i,t)} | \Delta^{(t-1)} \right] \pi^{(i)}(\phi, \theta) \right).
\]
Taking first-order conditions, we obtain:

\[
\frac{\partial \mathcal{L}(t)}{\partial q(n,t)} = \mathbb{E}\left[1_{\{I(t) \leq S(t)\}} \left( -\beta L_1^{(n,t)} + \beta \left(-L_1^{(n,t)} V_1(a(t), Q^{(c,t)}), \Delta^{(c,t)}\right) + V_2^{(n)}(a(t), Q^{(c,t)}, \Delta^{(c,t)}) \right) \right] + \lambda^{(n,t)} \mathbb{E}\left[\sum_{j=1}^{n} \beta_j L_j^{(n,t,j-1)} \right] \pi^{(n)} \left(\phi(I(t), S(t)), \mathbb{E}_{t-1} \left[R^{(t)} \right] \right) \Delta^{(c,t-1)} \right) = 0,
\]

\[
\frac{\partial \mathcal{L}(t)}{\partial p(n,t)} = \mathbb{E}\left[1_{\{I(t) \leq S(t)\}} \left( 1 + V_1(a(t), Q^{(c,t)}), \Delta^{(c,t)}\right) \right] \Delta^{(c,t-1)} \right) = 0,
\]

\[
\frac{\partial \mathcal{L}(t)}{\partial B(t)} = \mathbb{E}\left[1_{\{I(t) \leq S(t)\}} \left( -c'\left(B^{(t)}\right) + (1 - c'\left(B^{(t)}\right)) V_1(a(t), Q^{(c,t)}), \Delta^{(c,t)}\right) \right] - 1_{\{I(t) > S(t)\}} \Delta^{(c,t-1)} \right) = 0.
\]

The envelope theorem gives that:

\[
V_2^{(n)}(a(t), Q^{(c,t)}), \Delta^{(c,t)}) = -\beta \mathbb{E}\left[1_{\{I(t) \leq S(t)\}} L_2^{(n,t+1)} \left(1 + V_1(a^{(t+1)}, Q^{(c,t+1)}), \Delta^{(c,t+1)}\right) \right] + \Delta^{(c,t)} \right) = 0,
\]

\[
V_2^{(n)}(a(t+1), Q^{(c,t+1)}), \Delta^{(c,t+1)}) = -\beta \mathbb{E}\left[1_{\{I(t) \leq S(t)\}} L_3^{(n,t+2)} \left(1 + V_1(a^{(t+1)}, Q^{(c,t+2)}), \Delta^{(c,t+2)}\right) \right] + \Delta^{(c,t+1)} \right) = 0,
\]

\[
V_2^{(n)}(a^{(t+d_n-2)}, Q^{(c,t+d_n-2)}), \Delta^{(c,t+d_n-2)}) = -\beta \mathbb{E}\left[1_{\{I(t) \leq S(t+d_n-1)\}} L_{d_2}^{(n,t+d_n-1)} \left(1 + V_1(a^{(t+d_n-1)}, Q^{(c,t+d_n-1)}), \Delta^{(c,t+d_n-1)}\right) \right] + \Delta^{(c,t+d_n-2)} \right) = 0.
\]
Thus, taken together, we have:

\[
V_{2,1}^{(n)}(a^{(t)}, Q^{(t,t)}, \Delta^{(t,t)}) = -\beta \mathbb{E} \left[ I_{\{1^{(t,t)} \leq S^{(t,t)}\}} L_2^{(n,t+1)} \left( 1 + V_1(a^{(t+1)}, Q^{*(t,t+1)}, \Delta^{(t,t+1)}) \right) \right] \\
- \beta^2 \mathbb{E} \left[ I_{\{1^{(t,t+2)} \leq S^{(t,t+2)}\}} L_3^{(n,t+2)} \left( 1 + V_1(a^{(t+2)}, Q^{*(t,t+2)}, \Delta^{(t,t+2)}) \right) \right] \\
\vdots \\
- \beta^{d_n} \mathbb{E} \left[ I_{\{1^{(t,t+d_n-1)} \leq S^{(t,t+d_n-1)}\}} L_{d_n}^{(n,t+d_n-1)} \right] \\
\times \left( 1 + V_1(a^{*(t+d_n-1)}, Q^{*(t,t+d_n-1)}, \Delta^{(t,t+d_n-1)}) \right) \right] \\
= - \sum_{s=1}^{d_n-1} \beta^s \mathbb{E} \left[ I_{\{1^{(t,t+s)} \leq S^{(t,t+s)}\}} L_{s+1}^{(n,t+s)} \left( 1 + V_1(a^{*(t+s)}, Q^{*(t,t+s)}, \Delta^{(t,t+s)}) \right) \right]
\]

(16)

Now, since:

\[
\frac{\partial \phi}{\partial B^{(t)}} = \frac{\partial \phi}{\partial S^{(t)}} \frac{\partial S^{(t)}}{\partial B^{(t)}} = (1 + r)(1 - c'(B^{(t)})) \frac{\partial \phi}{\partial S^{(t)}} = (1 - c'(B^{(t)})) \frac{\partial \phi}{\partial p^{(n,t)}},
\]

by equating \( \sum_{i=1}^{N} \lambda^{(t,t)} \mathbb{E} \left[ \beta R^{(t,t)} \mid \Delta^{(t,t-1)} \right] \frac{\partial \phi}{\partial B^{(t)}} = \frac{\partial \phi}{\partial S^{(t)}} \cdot \frac{\partial S^{(t)}}{\partial B^{(t)}}, \)

in Equations (14) and (15), we obtain:

\[
\lambda^{(n,t)} = \frac{1}{1 - c'(B^{(t)})}, \quad \forall n = 1, 2, \ldots, N, \quad t = 1, 2, \ldots
\]

and, plugging this in (14), yields:

\[
\sum_{i=1}^{N} \mathbb{E} \left[ R^{(t,t)} \mid \Delta^{(t,t-1)} \right] \frac{\partial \phi}{\partial B^{(t)}} = 1 \frac{\partial \phi}{\partial S^{(t)}} = \frac{1}{1 - c'(B^{(t)})} - \mathbb{E} \left[ I_{\{1^{(t,t)} \leq S^{(t)}\}} \left( 1 + V_1(a^{(t)}, Q^{(t,t)}, \Delta^{(t,t)}) \right) \right].
\]

(17)

Let \( \rho \) be the risk measure associated with the risk metric \( \phi \), with adding-up property (Lemma 3.1):

\[
\sum_{i=1}^{N} \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\partial \rho(I^{(t,t)})}{\partial q^{(i,t-j)}} = \sum_{i=1}^{N} \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\partial}{\partial S^{(t)}} \phi(I^{(t,t)}, S^{(t)}) = S^{(t)}.
\]

Hence for every \( i, j \) and \( t \), we have:

\[
\frac{\partial \phi}{\partial q^{(i,t-j)}} = - \frac{\partial}{\partial S^{(t)}} \frac{\partial \rho(I^{(t,t)})}{\partial q^{(i,t-j)}}.
\]

(18)

With equations (16), (17) and (18), (13) becomes:

\[
\frac{\partial L^{(t)}}{\partial q^{(n,t)}} = -\beta \mathbb{E} \left[ I_{\{1^{(t,t)} \leq S^{(t)}\}} \left( L_1^{(n,t)} \left( 1 + V_1(a^{(t)}, Q^{(t,t)}, \Delta^{(t,t)}) \right) \right) \right] \\
- \sum_{s=1}^{d_n-1} \beta^s \mathbb{E} \left[ I_{\{1^{(t,t+s)} \leq S^{(t,t+s)}\}} L_{s+1}^{(n,t+s)} \left( 1 + V_1(a^{*(t+s)}, Q^{*(t,t+s)}, \Delta^{(t,t+s)}) \right) \right] \\
+ \frac{1}{1 - c'(B^{(t)})} \mathbb{E} \left[ \sum_{j=1}^{d_n-1} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(t,t-1)} \right] \times \left( \pi^{(n)} + \sum_{i=1}^{N} \pi_2^{(i)} R^{(i,t,t)} \right) \\
- \beta \mathbb{E} \left[ \sum_{i=1}^{N} \frac{1}{1 - c'(B^{(t)})} \mathbb{E} \left[ R^{(i,t,t)} \mid \Delta^{(t,t-1)} \right] \times \pi_1^{(i)} \frac{\partial \phi}{\partial S^{(t)}} \right] = 0,
\]

(19)
which by rearranging gives (8), and by Equation (14) the state weights \( w_t \) integrate to one.

**Proof of Proposition 5.1:** In contrast to the main text, we include a solvency constraint with risk measure \( \rho \) and consider the situation for \( n \) lines in the first part.

We derive expressions for marginal costs from Lagrangians of the right-hand size of the Bellman equation:

\[
\mathcal{L} = \mathbb{E} \left[ 1_{\{I \leq S\}} \left[ P - \beta I - \tau a - c_1(B) + \beta V(a', Q', M'_{j-1}) \right] + 1_{\{I > S\}}(-a - B) \right]
\]

\[
- \sum_{n=1}^{N} \lambda_n \left( p^{(n)} - p_n(a, Q, M_{j-1}, q', \ldots, q^{(N)}, P, B) \right)
\]

\[
- \eta (\beta \rho(I) - (a(1 - \tau) + B - c_1(B) + P)),
\]

where \( S = (a(1 - \tau) + B - c_1(B) + P)(1 + r) \). Denote \( I^{(i)}_q = \frac{\partial I}{\partial q^{(i)}} \), \( V_1 = \lim_{\Delta \to 0} \frac{V(a' + \Delta, Q', M'_{j-1}) - V(a', Q', M'_{j-1})}{\Delta} \), and \( V^{(i)}_2 = -\lim_{\Delta \to 0} \frac{V(a', Q'(q^{(i)} + \Delta), M'_{j-1}) - V(a', Q'(q^{(i)}), M'_{j-1})}{\Delta} \). The first order conditions for \( q, P, \) and \( B \) are:

\[
\frac{\partial \mathcal{L}}{\partial q^{(i)}} = \beta \mathbb{E} \left[ (-I^{(i)}_q(1 + V_1) - V^{(i)}_2) 1_{\{I \leq S\}} \right] + \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial q^{(i)}} - \eta \beta \left( \frac{\partial \rho}{\partial q^{(i)}} \right) = 0,
\]

(20)

\[
\frac{\partial \mathcal{L}}{\partial p^{(i)}} = \mathbb{E} \left[ (1 + V_1) 1_{\{I \leq S\}} \right] + f(S)((P - \tau a - c_1(B))(1 + r) - S + V) + (1 + r)f(S)(a + B)
\]

\[
- \lambda_i + \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial p^{(i)}} + \eta = 0,
\]

(21)

\[
\frac{\partial \mathcal{L}}{\partial B} = \mathbb{E} \left[ (-c_1'(B) + V_1(1 - c_1'(B))) 1_{\{I \leq S\}} - 1_{\{I > S\}} \right] + (1 - c_1'(B)) f(S)((P - \tau a - c_1(B))(1 + r)
\]

\[
- S + V) + (1 - c_1'(B))(1 + r)f(S)(a + B) + \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial B} + \eta(1 - c_1'(B)) = 0.
\]

(22)

From the first order conditions, we can solve for the vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T \). Denote

\[
E_q = \begin{pmatrix}
\text{an N x 1 vector of } \beta \mathbb{E} \left[ (-I^{(i)}_q(1 + V_1) - V^{(i)}_2) 1_{\{I \leq S\}} \right], i = 1, 2, \ldots, N
\end{pmatrix},
\]

\[
E_{PB} = \begin{pmatrix}
\text{an N x 1 vector of } (N - 1) \mathbb{E} \left[ (1 + V_1) 1_{\{I \leq S\}} \right]
\end{pmatrix},
\]
and

\[ E \left[ (-c'_1(B) + V_1(1 - c'_1(B)))I_{(I \leq S)} - I_{(I > S)} \right]. \]

Denote the following matrices:

\[ \Lambda_q = \begin{pmatrix}
\frac{\partial p_1}{\partial q^{(1)}} & \frac{\partial p_2}{\partial q^{(1)}} & \ldots & \frac{\partial p_N}{\partial q^{(1)}} \\
\frac{\partial p_1}{\partial q^{(2)}} & \ldots & \ldots & \ldots \\
\frac{\partial p_1}{\partial q^{(N)}} & \ldots & \ldots & \ldots \\
\end{pmatrix}, \quad \Lambda_{PB} = \begin{pmatrix}
1 & \frac{\partial p_2}{\partial p^{(1)}} & \ldots & \frac{\partial p_N}{\partial p^{(1)}} \\
\frac{\partial p_1}{\partial p^{(2)}} & 1 & \ldots & \ldots \\
\frac{\partial p_1}{\partial p^{(N-1)}} & \ldots & \ldots & 1 \\
\frac{\partial p_1}{\partial B} & \frac{\partial p_2}{\partial B} & \ldots & \frac{\partial p_N}{\partial B} \\
\end{pmatrix}, \]

\[ \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T. \]

Then solve for \( \Lambda = \Lambda_{PB}^{-1}E_{PB} \), then use \( \Lambda_q \Lambda = E_q \) to obtain capital allocation equation.

A special capital allocation equation in 2L2DY case can be derived as follows:

Denote \( X = L_1^{(1)} \). Note that \( X \) and \( I \), both univariate normal, can be represented as a bivariate normal distribution:

\[ \left( \begin{array}{c} X \\ I \end{array} \right) \sim N \left( \begin{array}{c} \mu_1^{(1)} \\ \mu_I \end{array} \right), \left( \begin{array}{cc} (\sigma_1^{(1)})^2 & \rho_{x,y} \sigma_1^{(1)} \sigma_I \\ \rho_{x,y} \sigma_1^{(1)} \sigma_I & \sigma_I^2 \end{array} \right) \right), \]

where \( \rho_{x,y} = \frac{q'(1)\sigma_1^{(1)} + q'(2)\rho \sigma_1^{(2)}}{\sigma_1^{(2)}}. \)

Let \( f(x, y) \) be the density of the bivariate normal of \( X \) and \( I \), \( f_{I|X}(y|x) \) be the conditional density of \( I \) given \( X \), \( f_I(y) \) be the marginal density of \( I \) and \( f_X(x) \) be the marginal density of \( X \). The conditional distribution of \( I \) given \( X \) is also univariate normal with mean and variance:

\[ \mu_{y|x} = \mu_I + \rho_{x,y} \sigma_I \frac{x - \mu_1^{(1)}}{\sigma_1^{(1)}}, \]

\[ = q'(1)x + q'(2) \frac{\mu_1^{(2)} + \rho \sigma_1^{(2)} x - \mu_1^{(1)}}{\sigma_1^{(1)}} + (f - 1)q^{(1)}L_1^{(1)} \]

\[ \sigma_{y|x}^2 = \sigma_I^2(1 - \rho_{x,y}^2) \]

\[ = (q'(2))^2(\sigma_1^{(2)})^2(1 - \rho^2) + (q^{(1)})^2 \sigma^2 L_1^{(1)}. \]
Their derivatives are:

\[
\begin{align*}
\frac{\partial \mu_{y|x}}{\partial q^{(1)}} &= x, \\
\frac{\partial \mu_{y|x}}{\partial q^{(2)}} &= \mu_2 + \rho \sigma_1 \frac{x - \mu_1}{\sigma_1}, \\
\frac{\partial \mu_{y|x}}{\partial p^{(n)}} &= \frac{\partial \mu_{y|x}}{\partial B} = 0, \ n = 1, 2 \\
\frac{\partial \sigma_{y|x}}{\partial q^{(1)}} &= \frac{\partial \sigma_{y|x}}{\partial p^{(n)}} = \frac{\partial \sigma_{y|x}}{\partial B} = 0, \ n = 1, 2 \\
\frac{\partial \sigma_{y|x}}{\partial q^{(2)}} &= \frac{q^{(2)}(\sigma_1^2)^2(1 - \rho^2)}{\sigma_{y|x}}.
\end{align*}
\]

Use the above equation to calculate the derivatives of default probabilities:

\[
\begin{align*}
P(I > S) &= 1 - \Phi_I(S), \\
\frac{\partial P(I > S)}{\partial p^{(1)}} &= -(1 + r) f_I(S), \\
\frac{\partial P(I > S)}{\partial B} &= (1 + r)(1 - c_1(B)) f_I(S), \\
\frac{\partial P(I > S)}{\partial q^{(1)}} &= \int_{-\infty}^{\infty} \frac{x}{\sigma_{y|x}} f_Z \left( \frac{S - \mu_{y|x}}{\sigma_{y|x}} \right) f_X(x) \, dx, \\
&= \int_{-\infty}^{\infty} \frac{x \, f(x, S)}{f_I(S)} \, dx \times f_I(S) \\
&= \mathbb{E} \left[ I^{(1)}_q | I = S \right] f_I(S) = \frac{\partial VaR_\psi(I)}{\partial q^{(1)}} f_I(S), \\
\frac{\partial P(I > S)}{\partial q^{(2)}} &= \int_{-\infty}^{\infty} \left( \frac{\mu_2 + \rho \sigma_1 \frac{x - \mu_1}{\sigma_1}}{\sigma_{y|x}} + \frac{S - \mu_{y|x}}{\sigma_{y|x}} \right) \frac{q^{(2)}(\sigma_1^2)^2(1 - \rho^2)}{\sigma_{y|x}} \\
&\times f_Z \left( \frac{S - \mu_{y|x}}{\sigma_{y|x}} \right) f_X(x) \, dx. \\
&= \mathbb{E} \left[ I^{(2)}_q | I = S \right] f_I(S) = \frac{\partial VaR_\psi(I)}{\partial q^{(2)}} f_I(S),
\end{align*}
\]
where $\Phi(x)$ and $f_Z(x)$ are the standard normal CDF and PDF. $f_I(S) = \Phi\left(\frac{S-\mu_I}{\sigma_I}\right)$ is the CDF of $I$ and $f_I(S) = \frac{1}{\sigma_I}f_Z\left(\frac{S-\mu_I}{\sigma_I}\right)$ is the PDF of $I$, both valued at $I = S$. $\psi = \mathbb{P}(I \leq S)$.

From equations (24) and (25), we have the following:

\[
\frac{\partial VaR_{\psi}(I)}{\partial q'(1)} = \int_{-\infty}^{\infty} \frac{x}{\sigma_y|x} f_Z\left(\frac{S-\mu_y|x}{\sigma_y|x}\right) f_X(x) dx f_I(S),
\]

\[
\frac{\partial VaR_{\psi}(I)}{\partial q'(2)} = \int_{-\infty}^{\infty} \left(\frac{\mu_2^{(2)} + \rho \sigma_1^{(2)} - \mu_1^{(1)}}{\sigma_y|x} + \frac{S-\mu_y|x}{\sigma_y|x} \frac{q'(2)(\sigma_2^{(2)})^2(1-\rho^2)}{(q(1))^2\sigma_2^2L_1^{(1)} + (q(2))^2(\sigma_2^{(2)})^2(1-\rho^2)}\right) f_Z\left(\frac{S-\mu_y|x}{\sigma_y|x}\right) f_X(x) dx f_I(S),
\]

which are used later in the marginal costs calculation.

From the premium functions:

\[
p_1 = \mathbb{E}\left[\beta R^{(1)}\right] \times \exp\left\{\alpha_1 - \delta_1 \mathbb{P}(I > S) - \gamma_1 \mathbb{E}[R]\right\}
\]

\[
p_2 = \mathbb{E}\left[\beta R^{(2)}\right] \times \exp\left\{\alpha_2 - \delta_2 \mathbb{P}(I > S) - \gamma_2 \mathbb{E}[R]\right\},
\]

we can derive the following derivatives for $p_i$ w.r.t. $q'(i)$, $p(i)$ and $B$.

\[
\frac{\partial p_1}{\partial q'(1)} = \mathbb{E}\left[\beta q'(1) L_1^{(1)} + e^{-2\rho} q'(1) L_2^{(1)}\right] \times \exp\left(\alpha_1 - \delta_1 \mathbb{P}(I > S) - \gamma_1 \mathbb{E}[R]\right) - p_1 \left(\delta_1 \mathbb{E}\left[I_q^{(1)}|I = S\right] f_I(S) + \gamma_1 \mathbb{E}\left[L_1^{(1)} + \beta L_2^{(1)}\right]\right),
\]

\[
\frac{\partial p_1}{\partial q'(2)} = -p_1 \left(\delta_1 \mathbb{E}\left[I_q^{(2)}|I = S\right] f_I(S) + \gamma_1 \mathbb{E}\left[L_1^{(2)}\right]\right),
\]

(26)
\[
\frac{\partial p_2}{\partial q^{(1)}} = -p_2 \left( \delta_2 \mathbb{E} \left[ I_1^{(1)} | I = S \right] f_I(S) + \gamma_2 \mathbb{E} \left[ L_1^{(1)} + \beta L_2^{(1)} \right] \right),
\]
\[
\frac{\partial p_2}{\partial q^{(2)}} = \mathbb{E} \left[ \beta R^{(2)} \right] \times \exp \left( \alpha_2 - \delta_2 \mathbb{P}(I > S) - \gamma_2 \mathbb{E}[R] \right)
- p_2 \left( \delta_2 \mathbb{E} \left[ I_1^{(2)} | I = S \right] f_I(S) + \gamma_2 \mathbb{E} \left[ L_1^{(2)} \right] \right),
\]
\[
\frac{\partial p_n}{\partial p^{(i)}} = p_n \delta_n (1 + r) f_I(S) \quad i = 1, 2,
\]
\[
\frac{\partial p_n}{\partial B} = p_n \delta_n (1 + r) (1 - c_1(B)) f_I(S).
\]

Then, derive the first order conditions for line 1 and 2:
\[
\frac{\partial \mathcal{L}}{\partial q^{(1)}} = \beta \mathbb{E} \left[ (-I_1^{(1)} (1 + V_1) - V_2) 1_{(I \leq S)} \right]
+ \lambda_1 \mathbb{E} \left[ \beta L_1^{(1)} + e^{-2rL_2^{(1)}} \right] \times \exp \left( \alpha_1 - \delta_1 \mathbb{P}(I > S) - \gamma_1 \mathbb{E}[R] \right)
- \lambda_1 p_1 \left( \delta_1 \mathbb{E} \left[ I_1^{(1)} | I = S \right] f_I(S) + \gamma_1 \mathbb{E} \left[ L_1^{(1)} + \beta L_2^{(1)} \right] \right)
- \lambda_2 p_2 \left( \delta_2 \mathbb{E} \left[ I_2^{(1)} | I = S \right] f_I(S) + \gamma_2 \mathbb{E} \left[ L_2^{(1)} + \beta L_2^{(1)} \right] \right)
- \eta \beta \left( \frac{\partial \rho}{\partial q^{(1)}} \right) = 0
\] (27)

\[
\frac{\partial \mathcal{L}}{\partial q^{(2)}} = \beta \mathbb{E} \left[ (-I_2^{(2)} (1 + V_1)) 1_{(I \leq S)} \right]
+ \lambda_2 \mathbb{E} \left[ \beta L_1^{(2)} \right] \times \exp \left( \alpha_2 - \delta_2 \mathbb{P}(I > S) - \gamma_2 \mathbb{E}[R] \right)
- \lambda_2 p_2 \left( \delta_2 \mathbb{E} \left[ I_2^{(2)} | I = S \right] f_I(S) + \gamma_2 \mathbb{E} \left[ L_1^{(2)} \right] \right)
- \lambda_1 p_1 \left( \delta_1 \mathbb{E} \left[ I_2^{(2)} | I = S \right] f_I(S) + \gamma_1 \mathbb{E} \left[ L_1^{(2)} \right] \right) - \eta \beta \left( \frac{\partial \rho}{\partial q^{(2)}} \right) = 0
\] (28)
\[
\begin{align*}
\frac{\partial L}{\partial p_{(i)}} &= \mathbb{E} \left[ (1 + V_1) \mathbb{I}_{(i \leq S)} \right] + f(x, S)((P - \tau a - c_1(B))(1 + r) - S + V) \\
&\quad + (1 + r)f(x, S)(a + B) - \lambda_i + (\lambda_1 p_1 \delta_1 + \lambda_2 p_2 \delta_2)(1 + r)f_I(S) + \eta = 0 \\
\Rightarrow \quad \mathbb{E} \left[ V_1 \mathbb{I}_{(i \leq S)} \right] + \mathbb{P}(I \leq S) + f(x, S)((P - \tau a - c_1(B))(1 + r) - S + V) \\
&\quad + (1 + r)f(x, S)(a + B) - \lambda_i + (\lambda_1 p_1 \delta_1 + \lambda_2 p_2 \delta_2)(1 + r)f_I(S) + \eta = 0
\end{align*}
\] (29)

\[
\begin{align*}
\frac{\partial L}{\partial B} &= \mathbb{E} \left[ (\lambda_1 p_1 \delta_1 + \lambda_2 p_2 \delta_2)(1 + r)(1 - c_1'(B))f_I(S) + \eta(1 - c_1'(B)) = 0 \\
\Rightarrow \quad \mathbb{E} \left[ V_1 \mathbb{I}_{(i \leq S)} \right] + \mathbb{P}(I \leq S) - \frac{1}{1 - c_1'(B)} + f(x, S)((P - \tau a - c_1(B))(1 + r) \\
&\quad - S + V) + (1 + r)f(x, S)(a + B) + (\lambda_1 p_1 \delta_1 + \lambda_2 p_2 \delta_2)(1 + r)f_I(S) + \eta = 0.
\end{align*}
\] (30)

Using equations (29) and (30), we can solve for \( \lambda_i \):

\[
\lambda_i = \frac{1}{1 - c_1'(B)}
\] (31)

Substitute \( \lambda_i \) back into equation (29), we can solve for \( \eta \)

\[
\eta = \mathbb{P}(I > S) + \frac{c_1'(B)}{1 - c_1'(B)} - \mathbb{E}[V_1 \mathbb{I}_{(i \leq S)}] - \frac{(p_1 \delta_1 + p_2 \delta_2)(1 + r)f_I(S)}{1 - c_1'(B)}
\] (32)

Without regulation constraint, \( \eta = 0 \), thus:

\[
\frac{(p_1 \delta_1 + p_2 \delta_2)(1 + r)f_I(S)}{1 - c_1'(B)} = \mathbb{P}(I > S) + \frac{c_1'(B)}{1 - c_1'(B)} - \mathbb{E}[V_1 \mathbb{I}_{(i \leq S)}]
\]

Our 2L2DY model assumes two lines have the same set of premium parameters, so from here on, \( \alpha = \alpha_1 = \alpha_2 \), \( \delta = \delta_1 = \delta_2 \), and \( \gamma = \gamma_1 = \gamma_2 \). Using equations (27) and (32), then on the right-hand-side add and subtract \( \mathbb{E}[\beta L_1^{(1)} \mathbb{I}_{(i \leq S)}] \) to obtain the marginal cost of risk for line 1, i.e.\]
equation (11) when $n = 1$:

$$
\frac{\left( \mathbb{E}[L_1^{(1)} + \beta L_2^{(1)}] \times \exp \left( \alpha - \delta \mathbb{P}(I > S) - \gamma \mathbb{E}[R] \right) \right)}{1 - c_1'(B)}
$$

$$
\times \mathbb{P}(I > S) + \frac{c_1'(B)}{1 - c_1'(B)} - \mathbb{E}[V_1 I_{\{I \leq S\}}]
$$

$$
\times \mathbb{E}\left[ (V_2 - \beta L_2^{(1)}) I_{\{I \leq S\}} \right]
$$

(33)

Each component, as integral below, can be calculated using numerical integration.

\begin{align*}
(i) & = \int_{-\infty}^{\infty} (1 + \beta(f - 1)) \times \phi \left( \frac{S - \mu y}{\sigma y} \right) f_X(x) \, dx \\
(ii) & = \int_{-\infty}^{\infty} \phi \left( \frac{S - \mu y}{\sigma y} \right) V_1(q, x) f_X(x) \, dx \\
(iii) & = \int_{-\infty}^{\infty} \phi \left( \frac{S - \mu y}{\sigma y} \right) V_1(q, x) f_X(x) \, dx \\
(iv) & = \int_{-\infty}^{\infty} \phi \left( \frac{S - \mu y}{\sigma y} \right) (V_2(a', x) - \beta(f - 1) x) f_X(x) \, dx
\end{align*}

Using equations (28) and (32) to obtain the marginal cost of risk for line 2:

$$
\frac{\left( \mathbb{E}[L_1^{(2)}] \times \exp \left( \alpha - \delta \mathbb{P}(I > S) - \gamma \mathbb{E}[R] \right) \right)}{1 - c_1'(B)}
$$

$$
\times \mathbb{P}(I > S) + \frac{c_1'(B)}{1 - c_1'(B)} - \mathbb{E}[V_1 I_{\{I \leq S\}}]
$$

(34)

While similar to (i) to (iii) in equation (33), evaluating each component in equation (34) is more complicated. First, since $L_1^{(2)}$ is correlated to $Y = I \mid L_1^{(1)}$ and $L_1^{(1)}$, calculating each expectation
requires solving a triple integral whenever those three random variables are involved. For example,

\[
(i) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{y}{q^{(2)}(x,y)} \right) x_2 f(x_2|y,x) \, dx_2 \right) f(y|x) \, dy \, f_X(x) \, dx
\]

Unfortunately, parts (i) to (iii) here neither resolves in closed form solution, nor can we solve by numerical integral technique. Instead, we use Monte-Carlo method and sample from multivariate normal distribution of \( L_1^{(1)}, L_1^{(2)}, L_1^{(1)} \). Then \( Y \) is the linear combination of three normal random variables and we can solve the expectations by taking the average of the samples that satisfy the criteria of the indicator. The rest of right-hand side of the equation can be evaluated similarly as in equation (33).

\[\Box\]

**B Details on Chain Ladder Assumptions**

In loss triangle depicted by Figure 3, \( L_1^{(n)} \) are losses paid in the previous accident year \( t-1 \). \( L_1^{(1)} \) is a realization included in the Markov structure, and therefore is a state variable in the optimal control problem. Line 2 has no development year, so \( L_1^{(2)} \) is irrelevant in the optimal control problem. \( L_1^{(1)}, L_1^{(2)}, \) and \( L_2^{(1)} \) are the losses to be paid in the current period \( t \), with \( L_1^{(1)} \) and \( L_1^{(2)} \) being paid losses for accident year \( t \) and \( L_2^{(1)} \) being the second development year paid loss for the previous accident year \( t-1 \) of line 1. Therefore, \( L_1^{(1)}, L_1^{(2)}, \) and \( L_2^{(1)} \) are the stochastic disturbances and have distributions subject to probability measures \( p(dL_1^{(1)}|L_1^{(1)}), p(dL_1^{(2)}|L_1^{(2)}) \) and \( p(dL_2^{(1)}|L_1^{(1)}) \). Meanwhile, \( L_2^{(1)} \) is the paid loss in the next period \( t+1 \), developed from \( L_1^{(1)} \) and therefore related to the current period’s premium in Line 1, \( L_2^{(1)} \) is also the stochastic disturbance subject to probability measures \( p(dL_2^{(1)}|L_1^{(1)}) \). We make the following assumptions for the properties of the probability measures:

Following **Chain-Ladder**, we assume:

\[
\mathbb{E}(L_2^{(1)}|L_1^{(1)}) = (f-1)L_1^{(1)}, \quad \mathbb{E}(L_2^{(1)}|L_1^{(1)}) = (f-1)L_1^{(1)}, \\
\mathbb{V}(L_2^{(1)}|L_1^{(1)}) = \sigma^2L_1^{(1)}, \quad \mathbb{V}(L_2^{(1)}|L_1^{(1)}) = \sigma^2L_1^{(1)}.
\]

We assume **conditional normality**:

\[
L_1^{(n)}|L_1^{(n)} \sim \mathcal{N}(\mu^{(n)}, (\sigma^{(n)})^2) \quad n = 1,2, \\
L_2^{(1)}|L_1^{(1)} \sim \mathcal{N}((f-1)L_1^{(1)}, \sigma^2L_1^{(1)}), \\
L_2^{(1)}|L_1^{(1)} \sim \mathcal{N}((f-1)L_1^{(1)}, \sigma^2L_1^{(1)}).
\]

We assume **a linear correlation** between lines:

\[
corr(L_1^{(1)}, L_1^{(2)}) = corr(L_1^{(1)}, L_1^{(2)}) = \rho.
\]

The chain-ladder property follows Mack (1993), whose model assumes \( M_{t,j-1}^{(n)} = C_{t,j-1}^{(n)} \). That is, the cumulative paid loss in each accident year is a Markov chain, with the ultimate development
year being the time horizon of the chain. $f$ and $\sigma^2$ are respectively the chain-ladder factor and its variance factor. Both factors can be estimated using a loss triangle. More precisely referring to Figure 2, using the past loss realization from accident year 1 to $t-2$, we can estimate the chain ladder factors following:

$$\hat{f} = \frac{\sum_{k=1}^{t-2} L_1^{(1,k)} + L_2^{(1,k+1)}}{\sum_{k=1}^{t-2} L_1^{(1,k)}},$$

$$\hat{\sigma}^2 = \frac{1}{t-3} \sum_{k=1}^{t-2} \left( L_1^{(1,k)} + L_2^{(1,k+1)} \right)^2 \left[ \frac{L_1^{(1,k)} + L_2^{(1,k+1)}}{L_1^{(1,k)}} - \hat{f} \right]^2.$$

As is conventional in this context, the indemnity is assumed to be proportional to the current-period exposures $q'(n)$ and last-period exposures $q(n)$. Hence, the indemnity random variable is specified as

$$I = q'(1)L_1^{(1)} + q'(2)L_1^{(2)} + q(1)L_2^{(1)}.$$ This linearity assumption entails that the marginal claim distribution is fixed, so that the loss distribution is homogeneous. In addition to linearity of indemnity, the conditional normality assumption ensures that $I$ is also normal, with conditional mean and variance:

$$\mu_I = \mathbb{E}(I|L_1^{(1)}) = q'(1)\mu_1^{(1)} + q'(2)\mu_1^{(2)} + (f-1)q(1)L_1^{(1)},$$

$$\sigma_I^2 = \mathbb{V}(I|L_1^{(1)}) = (q'(1))^2(\sigma_1^{(1)})^2 + (q'(2))^2(\sigma_1^{(2)})^2 + (q(1))^2\sigma_1^2L_1^{(1)} + 2q'(1)q'(2)\rho\sigma_1^{(1)}\sigma_1^{(2)}.$$

Again, this is in line with typical assumptions, and generalizations are possible.

We estimate model parameters for the accident year loss from $k = 1$ to $t-1$, using a simple regression with linear time trend:

$$L_1^{(n,k)} = \xi_0^{(n)} + \xi_1^{(n)}k + \epsilon_{k1}^{(n)} \sim \mathcal{N}(0, (\sigma_1^{(n)})^2),$$

so that:

$$\hat{\mu}_1^{(n)} = \hat{\xi}_0^{(n)} + \hat{\xi}_1^{(n)}t,$$

$$(\hat{\sigma}_1^{(n)})^2 = \frac{\sum_k (\hat{\epsilon}_{1}^{(n,k)})^2}{t-3},$$

$$\hat{\rho} = \text{corr}(\hat{\epsilon}_{1}^{(1,k)}, \hat{\epsilon}_{1}^{(2,k)}),$$

although the Bellman equation assumes identical distributions going forward.

### C Numerical Approach

To solve the Bellman equation, we rely on numerical methods. By our premium function assumption, $q'(1)$, $q'(2)$, and $B$ endogenously determine the sum of premiums $P$, and therefore these are the only choice variables. Note that the expectations of Equation (10) entail functions of $L_1^{(1)}$ and $I$, which renders the problem two-dimensional. To solve it, we use the numerical integration method from Tanskanen and Lukkarinen (2003).
First, let \( X = L_1^{(1)} \) and \( Y = I|L_1^{(1)} \). Note that \( X \) and \( Y \), both univariate normal, can be represented as a bivariate normal distribution:

\[
\left( \begin{array}{c}
X \\
Y
\end{array} \right) \sim \mathcal{N}\left( \left( \begin{array}{c}
\mu_1^{(1)} \\
\mu_I
\end{array} \right), \left( \begin{array}{cc}
(\sigma_1^{(1)})^2 & \rho_{x,y} \sigma_1^{(1)} \sigma_I \\
\rho_{x,y} \sigma_1^{(1)} \sigma_I & \sigma_I^2
\end{array} \right) \right),
\]

where \( \rho_{x,y} = \frac{q'(1)\sigma_1^{(1)} + q''(2)\rho_{1}^{(2)}}{\sigma^2} \). Hence, the conditional distribution of \( Y \) given \( X \) is also univariate normal with mean and variance

\[
\mu_{y|x} = \mu_I + \rho_{x,y} \sigma_I \frac{x - \mu_1^{(1)}}{\sigma_1^{(1)}},
\]

\[
\sigma_{y|x}^2 = \sigma_I^2 (1 - \rho_{x,y}^2),
\]

respectively.

Let \( f(x,y) \) be the density of the bivariate normal of \( X \) and \( Y \), \( f(y|x) \) be the conditional density of \( Y \) given \( X \), and \( f(x) \) be the marginal density of \( X \). Because of the nature of the value function is unknown, we need to use the value iteration method to solve the Bellman equation (10) on a discretized state-space. For 2L2DY model, there are three state variables: capital \( a \), last-period exposures on long-tailed line 1 \( q^{(1)} \), and last-period loss realizations on long-tailed line 1 \( L_1^{(1)} \).

Here are the steps of solving the Bellman equation using value iteration.

1. Pick grids for \( a = (a_1, a_2, \ldots, a_s) \), \( q^{(1)} = (q_1, q_2, \ldots, q_n) \), and \( L_1^{(1)} = (x_1, x_2, \ldots, x_p) \). Set \( V_0 = v_0(a, q^{(1)}, L_1^{(1)}) \), where \( v_0 \) is an arbitrary function.

2. Solve the optimization problem on the right hand side of the Bellman equation and get optimized state variables \( c^* \) and yield policy function \( c = u_1((a, q^{(1)}, L_1^{(1)}) ; c^*) \). Then obtain the next value function \( V_1((a, q^{(1)}, L_1^{(1)}); u_1) \) until \( V_j \) converges.

We can obtain a simplified Bellman equation for implementation:

\[
\beta \mathbb{E}\left\{ 1_{\{l \leq S\}} S - I \right\} = \int_{-\infty}^{S} \beta (S - y) f(y) \, dy
\]

\[
= \beta S \Phi_I(S) - \beta \int_{-\infty}^{S} y \frac{1}{\sqrt{2\pi}\sigma_I} e^{-\frac{(y-\mu_I)^2}{2\sigma_I^2}} \, dy
\]

\[
= \beta ((S - \mu_I) \Phi_I(S) + \sigma_I \Phi_I(S'),
\]

and

\[
\mathbb{E}\left\{ 1_{\{l \leq S\}} V(a', L_1^{(1)}) \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{S} V(a', L_1^{(1)}, x) f(x,y) \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{S} V(a', L_1^{(1)}, x) f(y|x) \, dy \right) f(x) \, dx,
\]

where \( \Phi \) and \( f_Z \), respectively, are the CDF and PDF of a standard normal distribution.

To solve the inner integral on a grid, we apply the Tanskanen and Lukkarinen (2003) method.
First, we interpolate on \( a \). We pick \((l+1)\) point grids for \( I \geq 0 \), say \((y_0, y_1, \ldots, y_l)\), with \( 0 = y_0 < y_1 < \cdots < y_l = S \), let \( \varphi_i = V(a'(y_i), q'(1), L_1^{(1)}) \).

For \( a'(y_i) \in (a_k, a_{k+1}) \), we approximately have by linear interpolation:

\[
\varphi_i = \frac{a_{k+1} - a'(y_i)}{a_{k+1} - a_k} V(a_k, q'(1), L_1^{(1)}) + \frac{a'(y_i) - a_k}{a_{k+1} - a_k} V(a_{k+1}, q'(1), L_1^{(1)}).
\]

If \( a'(y_i) > a_l \), we can extrapolate:

\[
\varphi_i = \frac{a'(y_l) - a_{l-1}}{a_l - a_{l-1}} V(a_l, q'(1), L_1^{(1)}) + \frac{a'(y_l) - a_l}{a_{l-1} - a_l} V(a_{l-1}, q'(1), L_1^{(1)}).
\]

The linear interpolation w.r.t. \( y \) is:

\[
V(a', q'(1), L_1^{(1)}) = \sum_{k=0}^{l-1} \left( \varphi_k + \frac{y - y_k}{y_{k+1} - y_k} (\varphi_{k+1} - \varphi_k) \right) I_{[y_k, y_{k+1}]}(y).
\]

We then break down the integral into sums:

\[
\int_{-\infty}^{S} V(a', q'(1), x) f(y|x) \, dy
= \sum_{k=0}^{l-1} \left[ \varphi_k - \frac{y_k (\varphi_{k+1} - \varphi_k)}{y_{k+1} - y_k} \right] \int_{y_k}^{y_{k+1}} f(y|x) \, dy + \left( \frac{\varphi_{k+1} - \varphi_k}{y_{k+1} - y_k} \right) \int_{y_k}^{y_{k+1}} y f(y|x) \, dy
\]

\[
= \sum_{k=0}^{l-1} \left\{ \left[ \varphi_k - \frac{y_k (\varphi_{k+1} - \varphi_k)}{y_{k+1} - y_k} \right] \left[ \Phi \left( \frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - \Phi \left( \frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right] \right. \\
+ \left( \frac{\varphi_{k+1} - \varphi_k}{y_{k+1} - y_k} \right) \left[ \mu_{y|x} \left[ \Phi \left( \frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - \Phi \left( \frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right] \right. \\
- \sigma_{y|x} \left( f_Z \left( \frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - f_Z \left( \frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right) \} \right\}
\]

\[
= h(x).
\]

Therefore the right-hand side of our Bellman equation can be written as

\[
\beta \left( (S - \mu_I) \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) + \sigma_I \phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right) + \int_{-\infty}^{\infty} \beta h(x) f(x) \, dx - a - B,
\]

which can be solved using a discretized grid of \( L_1^{(1)} \).

For our interpolation, we choose a \( m+1 \) point equally spaced grid on \([\mu_1^{(1)} - 2\sigma_1^{(1)}, \mu_1^{(1)} + 2\sigma_1^{(1)}]\), say \((x_0, x_1, \ldots, x_m)\). In our case the grid size is 26. Use the trapezoidal rule to break the integral down into sums, say \( F(x) = \beta (g(x) + h(x)) f(x) \), then the integral becomes:

\[
\int_{-\infty}^{\infty} F(x) \, dx = \frac{x_m - x_0}{2m} (F(x_0) + 2F(x_1) + \cdots + 2F(x_{m-1}) + F(x_m)).
\]
Hence we successfully convert double integrals into sums and therefore significantly reduce the computation time without compromising the accuracy. The value iteration is implemented and run in Julia. The optimization is executed using Julia’s NLopt package. The value function is defined on a 21 x 21 x 3 discretized grid (with 21 grid points on $a$ and $q^{(1)}$). We ran the program for 80 iterations and the value function converges for both value function and choice variables.

The total runtime was about 100,000s, on a Intel I7 dual-core CPU. In this specific numerical task, Julia is six times faster than the popular high-level language such as R and MatLab. In particular, Julia is much faster with loops, which is heavily found in the iterations and numerical integrals. According to Julia language developers, Julia is a high-performance language suitable for dynamic programming and its syntax is easily adapted from R or Matlab. Julia’s high efficiency helps shorten the runtime from one week that would have taken on R, to just under one day.

Compared to previous models without considering DY, which only has one state variable capital, the 2L2DY model suffers from “the curse of dimensionality”. As the general $nLjDY$ model has hundreds, million, or even trillion times more states, each iteration of value function would take proportional more time to complete, resulting the value iteration to finish in months or even years. Solving this high-dimensional problem seems infeasible even five years ago, but thanks to the power of modern day computing, it is feasible under proper assumptions. We start with solving 2L2DY, which has the least dimension in the general $nLjDY$ model. In the future, we will continue to refine the algorithm and implement parallel computing to further shorten the running time.

D Additional Figures

![Figure 6: Convergence of value function and choice variables](image-url)